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CONTRIBUTION TO THE THEORY OF HEAT TRANSFER ACROSS A TURBULENT BOUNDARY LAYER

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Abstract—It is shown that certain regularities exist in the exact solutions of the partial differential equation of uniform-property heat transfer published by Gardner and Kestin [3] and Smith and Shah [4]. These regularities permit the development of approximate formulae for the Stanton number which are probably as reliable, as means of predicting heat transfer, as are the solutions based on numerical integration.

The solutions are generalized so as to hold for the case in which the "turbulent Prandtl number" is a constant differing from unity, and it is argued that a value in the neighbourhood of 0·887 should be used in future work. A discussion is presented of the way in which the theory can be extended to: rough walls; mass transfer at a finite rate; and non-uniform fluid properties.

NOMENCLATURE

Number in parentheses denotes the equation of first appearance.

- A_q, wall temperature downstream of line heat sink (28);
- c, specific heat at constant pressure (Btu/lb degF) (4);
- c_f, local drag coefficient (16);
- E, a constant (20);
- k, (laminar) thermal conductivity (Btu/ft h degF);
- k_t, total thermal conductivity (Btu/ft h degF) (4);
- K, a constant (20);
- $\dot{m}''$, local mass-transfer flux (lb/ft²h);
- N_{Pr}, (laminar) Prandtl number ($=c\mu/k$) (10);
- $N_{Pr,t}$, turbulent Prandtl number (49);
- $N_{Re,x}$, Reynolds number based on x (35);
- N_{RF}, recovery factor (64);
- N_{St}, Stanton number (11);
- P, "extra thermal resistance" of the laminar sub-layer (27);
- $\dot{q}'$, strength of line heat sink (Btu/ft h) (28);
- $\dot{q}''$, heat flux to the wall (Btu/ft²h) (19);
- R, contribution to recovery factor made by the laminar sub-layer (64);
- S_T, non-dimensional heat-transfer coefficient (13);
- S_q, non-dimensional heat-transfer coefficient (16);
- T, temperature (°F) (1);
- t^+, non-dimensional temperature (31);
- u, velocity along the wall (ft/h) (5);
- u^+, non-dimensional velocity (1);
- x, distance along a streamline (ft) (2);
- x^+, non-dimensional velocity along a streamline (1);
- y, distance from and normal to the wall (ft);
- y^+, non-dimensional distance from and normal to the wall (44);
- y_r, nominal height of roughness element (ft) (62);
- y_r^+, non-dimensional height of roughness element (62);
- α, heat-transfer coefficient (Btu/ft²h degF) (12);
- ϵ_u^+, non-dimensional total viscosity (1);
- ϵ_h^+, non-dimensional total thermal conductivity (1);
- μ, (laminar) dynamic viscosity (lb/ft h) (3);
- μ_t, total viscosity (lb/ft h) (3);
- ρ, density (lb/ft³) (2);
- τ_S, shear stress at wall (lb/ft h²) (2);

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ξ, non-dimensional distance from wall (7);
ϕ, turbulent contribution to ϵ_u^+ (48).

Subscripts
G, main stream;
S, wall;
1, value pertaining to $N_{Pr} = N_{Pr,t}$; except in the case of u_1^+ where the "join of the laminar and turbulent layers" is referred to.

1. INTRODUCTION

1.1 *Previous work*

Several papers have appeared recently which attempt to develop an exact theory of heat transfer through the uniform-property turbulent boundary layer. They are distinguished from earlier work in this field by the fact that they rest on exact solutions of a partial differential equation for convective heat transfer in a flow field characterized by the so-called universal turbulent velocity profile, i.e. the "law of the wall". Unlike the older Couette-flow analyses, in which variations of temperature in the flow direction are neglected, they can, therefore, be expected to give precise agreement with experiment provided only that the thermal boundary layer is appreciably thinner than the velocity boundary layer and that the true properties of the latter are incorporated.

The relevant partial differential equation was derived in [1], where a solution was presented for a Prandtl number of unity and a prescribed wall-temperature distribution; the solution was obtained by means of an analogue computer. Kestin and Persen [2] obtained a more exact solution for this problem, using a digital computer; their work was extended to non-unity Prandtl numbers by Gardner and Kestin [3]. Smith and Shah [4] obtained digital-computer solutions to the same differential equation for three Prandtl numbers for the case in which the heat flux is specified rather than the wall temperature.

In all the above papers, the specification of the properties of the flow field was that of [1], with minor variations in the constants. It is probable, as will be argued later in the present paper, that this specification is inadequate, particularly in its implication that the effective Prandtl number in the turbulent part of the boundary layer is unity. It is, therefore, important to distinguish the main structure of the theory, which can be regarded as exact, from the particular formulations of the flow-field properties, which will probably require amendment in the light of more complete experimental information.

1.2 *Purposes of the present paper*

The solutions to the partial differential equation which have been obtained permit a number of interesting conclusions to be drawn. Regularities appear in them which permit new solutions to be generated without extensive further computations; these regularities incidentally provide a certain degree of rehabilitation to the Couette-flow analysis which the theory was designed to replace. When, however, the implications of the theory for flat-plate heat transfer are compared with experimental data, the assumption that the effective kinematic viscosity and thermal diffusivity in the turbulent region are equal ceases to be tenable. Fortunately the abandonment of this assumption does not entail the discarding of the computations already carried out: they can be made applicable to any uniform value of the turbulent Prandtl number.

It is the purpose of the present paper to point out and explain the regularities referred to, and to simplify, extend and generalize the existing solutions to the partial differential equation. In addition, it is proposed to discuss the way in which the theory can be extended to heat transfer from rough surfaces, to mass transfer at a finite rate, and to flows with non-uniform fluid properties.

2. SIMPLIFICATION AND EXTENSION OF THE EXISTING SOLUTIONS

2.1 *Nature of the mathematical problems and their solutions.*

The differential equation. The temperature T in a uniform-property universal turbulent boundary layer, whether two-dimensional or axi-symmetrical, has been shown in [1] to be governed by the partial differential equation:

$$\frac{\partial T}{\partial x^+} = \frac{1}{u^+ \epsilon_u^+} \frac{\partial}{\partial u^+} \left(\frac{\epsilon_h^+}{\epsilon_u^+} \frac{\partial T}{\partial u^+} \right) \tag{1}$$

where

$$x^+ \equiv \int_0^x (\tau_S \rho)^{1/2} \mu^{-1} \, dx \qquad (2)$$

$$\epsilon_u^+ \equiv \mu_t/\mu = \epsilon_u^+ (u^+) \qquad (3)$$

$$\epsilon_h^+ \equiv k_t/(c\mu) = \epsilon_h^+ (u^+) \qquad (4)$$

$$\mu^+ \equiv u/(\tau_S/\rho)^{1/2} \qquad (5)$$

and

τ_S, shear stress at wall;
ρ, fluid density;
μ, fluid viscosity;
x, distance along wall in flow direction;
μ_t, total (i.e. laminar plus turbulent) viscosity;
k_t, total (i.e. laminar plus turbulent) thermal conductivity;
c, specific heat of fluid at constant pressure;
u, time-mean velocity in the x direction.

The differential equation may be re-written, for convenient numerical integration, in terms of a new independent variable ξ, as:

$$\frac{\partial T}{\partial x^+} = \frac{1}{u^+ \epsilon_h^+} \frac{\partial^2 T}{\partial \xi^2} \qquad (6)$$

where

$$\xi \equiv \int_0^{u^+} \frac{\epsilon_u^+}{\epsilon_h^+} \, du^+. \qquad (7)$$

Of course, u^+ and ϵ_h^+ must now be regarded as functions of ξ.

2.2 *The two basic problems*

The step in wall temperature. In [1], [2] and [3], the problem solved was that in which the wall temperature equalled the stream temperature T_G upstream of the plane $x = 0$, but was held at the uniform value T_S downstream of this plane. The initial and boundary conditions were therefore:

$$\left.\begin{array}{l} x^+ = 0, \; u^+ \text{ (or } \xi) \geq 0 \\ \text{all } x^+, \quad u^+ \text{ (or } \xi) \to \infty \end{array}\right\} : T = T_G \qquad (8)$$

$$x^+ > 0, \; u^+ \text{ (or } \xi) = 0 \quad : T = T_S. \qquad (9)$$

Solution of the equation then yields $T(x^+, u^+)$, and therefore the temperature gradient at the wall as a function of x^+. This gradient is related to the local heat flux and other quantities as follows:

$$N_{Pr} \frac{(\partial T/\partial \xi)_S}{T_G - T_S} = \frac{(\partial T/\partial u^+)_S}{T_G - T_S} \qquad (10)$$

$$= \frac{N_{Pr} \, N_{St}}{(c_f/2)^{1/2}} \qquad (11)$$

$$= N_{Pr} \frac{\alpha}{c \, (\tau_S \rho)^{1/2}} \qquad (12)$$

$$\equiv S_T, \text{ say} \qquad (13)$$

where N_{St} and N_{Pr} are respectively the local Stanton number and the laminar Prandtl number, c_f is the local drag coefficient, and α is the local heat-transfer coefficient.

The function $S_T\,(x^+)$ is important because it can be used, by the employment of well-known superposition techniques, for the computation of the heat-flux distribution with an *arbitrary* wall-temperature distribution.

The step in heat flux through the wall. In [4] solutions to the differential equation were obtained for the situation in which the non-dimensional heat-flux was zero upstream of the plane $x^+ = 0$, but had a uniform value downstream of this plane. The initial and boundary conditions were therefore:

$$\left.\begin{array}{l} x^+ = 0, \; u^+ \text{ (or } \xi) \geq 0 \\ \text{all } x^+, \quad u^+ \text{ (or } \xi) \to \infty \end{array}\right\} T = T_G \qquad (14)$$

$$x^+ > 0, \; u^+ \text{ (or } \xi) = 0 : \left(\frac{\partial T}{\partial \xi}\right)_S = \text{const} \qquad (15)$$

Solution of this problem yields $T\,(x^+, u^+)$ and therefore the wall temperature T_S as a function of x^+. The solution was expressed by Smith and Shah [4] in terms of $N_{St}/\sqrt{(c_f/2)}$. For uniformity with the practice of [2] and [3], in the present paper solutions to this problem will be expressed in terms of S_q, which is related to other variables by:

$$S_q \equiv \frac{N_{Pr} \, N_{St}}{(c_f/2)^{1/2}} \qquad (16)$$

$$= N_{Pr} \frac{(\partial T/\partial \xi)_S}{T_G - T_S} \qquad (17)$$

$$= \frac{(\partial T/\partial u^+)_S}{T_G - T_S} \qquad (18)$$

$$= \frac{\dot{q}''/(T_G - T_S)}{c \, (\tau_S \rho)^{1/2}} \qquad (19)$$

where $\dot{q}''$ is the local heat flux *to* the interface. Of course, S_q is related to Stanton number, etc., in precisely the same way as is S_T; the different subscript is merely a reminder of the different boundary condition (heat-flux rather than temperature).

The S_q function is important because it can be used, by the employment of well-known superposition techniques, for the computation of the wall-temperature distribution resulting from an arbitrary heat-flux distribution.

The functions ϵ_u^+ *and* ϵ_h^+. The non-dimensional total viscosity and thermal conductivity are postulated to be functions of u^+ and the laminar Prandtl number alone. For want of better information, the following relations were adopted in [1], [2], [3] and [4]:

$$\epsilon_u^+ = 1 + (K/E)\,[e^{Ku^+} - 1 - Ku^+ - (Ku^+)^2/2! - (Ku^+)^3/3!] \quad (20)$$

and

$$\epsilon_h^+ = (1/N_{Pr}) + (K/E)\,[e^{Ku^+} - 1 - Ku^+ - (Ku^+)^2/2! - (Ku^+)^3/3!] \quad (21)$$

where K and E are constants ($K = 0{\cdot}4$ and $E = 9{\cdot}025$ were adopted in [2] and [3]; $K = 0{\cdot}407$ and $E = 10{\cdot}09$ were adopted in [1] and [4]).

Inspection of these expressions reveals that close to the wall, i.e. at small u^+, ϵ_u^+ is equal to N_{Pr} times ϵ_h^+; far from the wall, i.e. at large u^+, ϵ_u^+ and ϵ_h^+ are equal. Whereas (20) will be adopted throughout the present paper, we shall find reason to modify (21) below. ϵ_u^+/ϵ_h is, of course, the "total Prandtl number".

2.3 *The existing solutions*

The S_T function computed in references [2] and [3] is plotted in Fig. 1 (full lines). [N.B. There the abscissa is $x^+/N_{Pr,t}$ rather than x^+ and the parameter $N_{Pr}/N_{Pr,t}$ rather than N_{Pr}. The present section of the report is written as though the $N_{Pr,t}$'s were absent (i.e. equal to unity), as indeed they were in the work of references [2] and [3] The reason for the appearance of the $N_{Pr,t}$'s is explained in section 3.3.]

At low values of x^+, the function has the asymptotic form:

$$S_T \to 0{\cdot}53835\,(x^+/N_{Pr})^{1/3}. \quad (22)$$

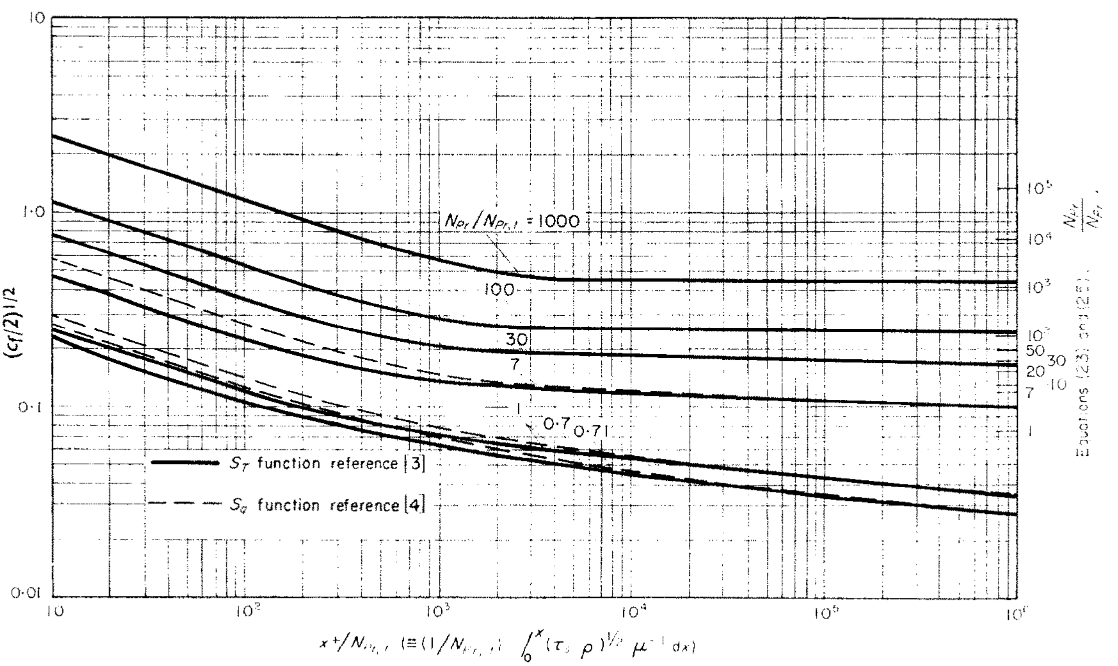

FIG. 1. Numerically calculated solutions of equation (1).

Table 1. Values of $\xi - u^-$ obtained from the exact solutions

Data from Gardner and Kestin [3]

N_{Pr}	0·71		7·0			30·0			100·0			1000		
$13·4\,(N_{Pr}-1)N_{Pr}^{-1/4}$	−4·233		49·429			166·044			419·508			2380·512		
u^-	ξ	$\xi - u^+$	u^+	ξ	$\xi - u^+$	u^+	ξ	$\xi - u^-$	u^+	ξ	$\xi - u^+$	u^+	ξ	$\xi - u^+$
0·565	0·4011	−0·1639	0·2290	1·603	1·374	0·1340	4·020	3·886	0·1400	14·00	13·8600	0·0800	80·00	79·92
1·125	0·7988	−0·3262	0·4570	3·199	2·742	0·2660	7·980	7·714	0·2800	28·00	27·7200	0·1600	160·0	159·84
1·690	1·200	−0·490	0·6860	4·802	4·116	0·4000	12·00	11·800	0·4200	42·00	41·5800	0·2400	240·0	239·76
2·255	1·601	−0·654	0·9140	6·398	5·484	0·5340	16·02	15·486	0·5600	55·99	55·4300	0·3200	320·0	319·68
2·815	1·999	−0·816	1·143	8·000	6·857	0·6660	19·98	19·314	0·7000	69·98	69·28	0·4000	399·9	399·5
3·380	2·401	−0·979	1·372	9·602	8·230	0·8000	24·00	23·200	0·8400	83·86	83·02	0·4800	479·8	479·32
3·940	2·800	−1·140	1·601	11·20	9·599	0·9340	28·01	27·076	0·9800	97·91	96·93	0·5600	559·5	558·94
4·500	3·200	−1·300	1·830	12·80	10·970	1·068	32·03	30·962	1·122	112·0	110·878	0·6420	641·0	640·358
5·055	3·598	−1·457	2·060	14·40	12·340	1·200	35·98	34·780	1·264	126·1	124·836	0·7220	720·1	719·378
5·610	3·999	−1·611	2·290	16·00	13·710	1·334	39·98	38·646	1·406	140·0	138·594	0·8040	800·7	799·896
6·160	4·400	−1·760	2·521	17·60	15·079	1·468	43·98	42·512	1·550	154·1	152·55	0·8860	880·6	879·714
6·700	4·799	−1·901	2·754	19·20	16·446	1·604	48·02	46·416	1·694	168·0	166·306	0·9680	959·7	958·732
7·235	5·199	−2·036	2·988	20·80	17·812	1·738	51·99	50·252	1·842	182·0	180·158	1·052	1039·0	1037·948
7·760	5·600	−2·160	3·224	22·40	19·176	1·874	56·01	54·136	1·992	196·0	194·008	1·138	1119·0	1117·862
8·275	6·001	−2·274	3·464	24·00	20·536	2·010	59·99	57·980	2·146	209·9	207·754	1·226	1199·0	1197·774
8·775	6·399	−2·376	3·707	25·60	21·893	2·148	64·01	61·862	2·226	217·0	214·774	1·318	1280·0	1278·682
9·265	6·798	−2·467	3·955	27·20	23·245	2·286	67·99	65·704	2·306	224·0	221·694	1·412	1360·0	1358·588
9·745	7·198	−2·547	4·209	28·80	24·591	2·426	71·99	69·564	2·388	231·0	228·612	1·512	1440·0	1438·488
10·22	7·599	−2·621	4·471	30·40	25·929	2·714	80·02	77·306	2·472	238·0	235·528	1·618	1520·0	1518·382
10·68	7·999	−2·681	4·743	32·00	27·257	2·860	83·99	81·130	2·558	245·0	242·442	1·732	1600·0	1598·268
11·13	8·398	−2·732	5·027	33·60	28·573	3·012	88·02	85·008	2·648	252·1	249·452	1·794	1640·0	1638·206
12·01	9·198	−2·812	5·327	35·20	29·873	3·166	91·99	88·824	2·738	258·9	256·162	1·858	1680·0	1678·142
12·87	10·00	−2·87	5·647	36·80	31·153	3·326	95·99	92·664	2·834	266·0	263·166	1·926	1720·0	1718·074
13·71	10·80	−2·91	5·993	38·40	32·407	3·492	99·98	96·488	2·932	272·9	269·968	2·000	1760·0	1758·000
14·54	11·60	−2·94	6·373	40·00	33·627	3·666	104·00	100·334	3·036	280·0	276·964	2·078	1800·0	1797·922
15·40	12·40	−3·00	6·799	41·60	34·801	3·85	108·0	104·15	3·146	287·0	283·854	2·164	1840·0	1837·836
16·17	13·20	−2·97	7·284	43·20	35·916	4·046	112·0	107·954	3·260	294·0	290·740	2·258	1880·0	1877·742
16·98	14·00	−2·98	7·851	44·80	36·949	4·256	116·0	111·744	3·384	301·0	297·616	2·364	1920·0	1917·636
17·79	14·80	−2·99	8·527	46·40	37·873	4·486	120·0	115·514	3·516	308·0	304·484	2·484	1960·0	1957·516
18·60	15·60	−3·00	9·344	48·00	38·656	4·742	124·0	119·258	3·660	315·0	311·340	2·622	2000·0	1997·378
19·40	16·40	−3·00	10·33	49·60	39·27	5·032	128·0	122·968	3·818	322·0	318·182	2·790	2040·0	2037·21
20·20	17·20	−3·00	11·50	51·20	39·70	5·370	132·0	126·630	3·994	329·0	325·006	2·998	2080·0	2077·002
21·01	18·00	−3·01	12·83	52·80	39·97	5·778	136·0	130·222	4·194	336·0	331·806	3·276	2120·0	2116·724
22·21	19·20	−3·01	14·27	54·40	40·13	6·298	140·0	133·702	4·426	343·0	338·574	3·686	2160·0	2156·314
23·01	20·00	−3·01	15·79	56·00	40·21	7·006	144·0	136·994	4·706	350·0	345·294	3·826	2170·0	2166·174
24·21	21·20	−3·01	17·35	57·60	40·25	8·066	148·0	139·934	5·054	357·0	351·946	3·990	2180·0	2176·01
25·41	22·40	−3·01	18·92	59·20	40·28	9·834	152·0	142·166	5·514	364·0	358·486	4·186	2190·0	2185·814
26·21	23·20	−3·01	20·51	60·80	40·29	12·71	156·0	143·290	6·186	371·0	364·814	4·430	2200·0	2195·57
27·41	24·40	−3·01	22·11	62·40	40·29	16·39	160·0	143·610	7·334	378·0	370·666	4·746	2210·0	2205·254
28·61	25·60	−3·01	23·71	64·00	40·29	20·34	164·0	143·660	9·918	385·0	375·082	5·186	2220·0	2214·814
29·81	26·80	−3·01	25·30	65·60	40·30	24·33	168·0	143·67	15·53	392·0	376·47	5·878	2230·0	2224·122
31·01	28·00	−3·01	26·90	67·20	40·30	28·33	172·0	143·67	22·43	399·0	376·57	7·252	2240·0	2232·748
32·21	29·20	−3·01	28·50	68·80	40·30	32·33	176·0	143·67	29·43	406·0	376·57	11·65	2250·0	2238·35
33·41	30·40	−3·01	30·10	70·40	40·30	36·33	180·0	143·67	36·43	413·0	376·57	21·18	2260·0	2238·82
34·61	31·60	−3·01	33·30	73·60	40·30	40·33	184·0	143·67	43·43	420·0	376·57	31·18	2270·0	2238·82
35·81	32·80	−3·01	36·50	76·80	40·30	44·33	188·0	143·67	50·43	427·0	376·57	41·18	2280·0	2238·82

Data from Smith and Shah [4]

N_{Pr}	0·7			7·0		
$13·4\,(N_{Pr}-1)N_{Pr}^{-1/4}$	−4·395			49·429		
	u^+	$\xi - u^-$	ξ	u^-	ξ	$\xi - u^-$
	1	0·7	−0·3	1	6·999	5·999
	2	1·4	−0·6	2	13·99	11·99
	3	2·101	−0·899	3	20·88	17·88
	4	2·803	−1·197	4	27·50	23·50
	5	3·509	−1·491	5	33·48	28·48
	6	4·223	−1·777	6	38·48	32·48
	7	4·953	−2·047	7	42·35	35·35
	8	5·702	−2·298	8	45·25	37·25
	9	6·496	−2·504	9	47·44	38·44
	10	7·322	−2·678	10	49·18	39·18
	11	8·189	−2·811	11	50·62	39·62
	12	9·092	−2·908	12	51·90	39·90
	13	10·02	−2·98	13	53·07	40·07
	14	10·98	−3·02	14	54·18	40·18
	15	11·95	−3·05	15	55·25	40·25
	16	12·92	−3·08	16	56·29	40·29
	17	13·91	−3·09	17	57·32	40·32
	18	14·90	−3·10	18	58·34	40·34
	19	15·90	−3·10	19	59·35	40·35
	20	16·89	−3·11	20	60·36	40·36
	21	17·89	−3·11	21	61·36	40·36
	22	18·88	−3·12	22	62·37	40·37
	23	19·87	−3·13	23	63·37	40·37
	24	20·87	−3·13	24	64·37	40·37
	25	21·87	−3·13	25	65·38	40·38
	26	22·87	−3·13	26	66·38	40·38
	27	23·87	−3·13	27	67·38	40·38
	28	24·87	−3·13	28	68·38	40·38
	29	25·87	−3·13	29	69·38	40·38
	30	26·87	−3·13	30	70·38	40·38
	31	27·87	−3·13	31	71·38	40·38
	32	28·87	−3·13	32	72·38	40·38
	33	29·87	−3·13	33	73·38	40·38
	34	30·87	−3·13	34	74·38	40·38
	35	31·87	−3·13	35	75·38	40·38
	36	32·87	−3·13	36	76·38	40·38
	37	33·87	−3·13	37	77·38	40·38

Data from Mills [8]

N_{Pr}	0·01		0·10		0·7		1·0		3·333		10·0		100		1000	
$13·4\,(N_{Pr}-1)N_{Pr}^{-1/4}$	−41·951		−21·446		−4·395		0·0		23·137		67·818		419·508		2380·512	
u^+	ξ	$\xi - u^-$	ξ	$\xi - u^-$	ξ	$\xi - u^+$	ξ	$\xi - u^-$	ξ	$\xi - u^+$	ξ	$\xi - u^+$	ξ	$\xi - u^+$	ξ	$\xi - u^+$
18·0	1·1809	−16·8191	5·5274	−12·4726	14·901	−3·099	18·0	0	36·156	18·156	74·38	56·38	394·86	376·86	2272·8	2254·8
20·0	2·1250	−17·8750	7·3385	−12·6615	16·891	−3·109	20·0	0	38·172	18·172	76·41	56·41	396·89	376·89	2274·8	2254·8
22·0	3·4643	−18·5357	9·2513	−12·7487	18·887	−3·113	22·0	0	40·179	18·179	78·43	56·43	398·91	376·91	2276·8	2254·8
24·0	5·1062	−18·8938	11·212	−12·788	20·885	−3·115	24·0	0	42·182	18·182	80·45	56·45	400·93	376·93	2278·8	2254·8
26·0	6·9301	−19·0699	13·195	−12·805	22·884	−3·116	26·0	0	44·183	18·183	82·46	56·46	402·93	376·93	2280·8	2254·8
28·0	8·8482	−19·1518	15·187	−12·813	24·884	−3·116	28·0	0	46·183	18·183	84·46	56·46	404·93	376·93	2282·8	2254·8
30·0	10·811	−19·189	17·183	−12·817	26·884	−3·116	30·0	0	48·184	18·184	86·46	56·46	406·93	376·93	2284·8	2254·8

Table 2. Data for S_T and S_1 computed in references [3] and [4], and $N_{Pr}/S - 1/S_1$ computed therefrom generalized in accordance with the recommendations of section 3.3 of the text. In references [3] and [4] $N_{Pr,1}$ was taken as unity

	Data from Gardner and Kestin [3]																		Data from Smith and Shah [4]								
	$N_{Pr}/N_{Pr,1}$ = 0·71			$N_{Pr}/N_{Pr,1}$ = 1·0			$N_{Pr}/N_{Pr,1}$ = 7·0			$N_{Pr}/N_{Pr,1}$ = 30·0			$N_{Pr}/N_{Pr,1}$ = 100·0			$N_{Pr}/N_{Pr,1}$ = 1000·0			$N_{Pr}/N_{Pr,1}$ = 0·7			$N_{Pr}/N_{Pr,1}$ = 1·0			$N_{Pr}/N_{Pr,1}$ = 7·0		
$x^*/N_{Pr,1}$	S_T	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{T,1}}$	S_T	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{T,1}}$	S_T	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{T,1}}$	S_T	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{T,1}}$	S_T	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{T,1}}$	S_T	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{T,1}}$	S_q	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{q,1}}$	S_q	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{q,1}}$	S_q	$(c_f/2)^{1/2}/N_{St}$	$\frac{(c_f/2)^{1/2}}{N_{St}} - \frac{1}{S_{q,1}}$
1 × 10	0·2228	3·187	−0·817	0·2498	4·003	0	0·4779	14·65	10·64	0·7763	38·64	34·64	1·1598	86·22	82·22	2·4984	400·5	396·5	0·2667	2·624	−0·749	0·2965	3·373	0	0·5729	12·22	8·65
2 × 10	0·1771	4·009	−1·024	0·1987	5·033	0	0·3793	18·46	13·42	0·6158	48·72	43·68	0·9203	108·66	103·63	1·9807	504·9	499·8	0·2202	3·331	−0·885	0·2372	4·215	0	0·4534	15·44	11·22
3 × 10	0·1554	4·569	−1·172	0·1742	5·741	0	0·3321	21·08	15·34	0·5387	55·69	49·95	0·8047	124·27	118·53	1·7317	577·5	571·7	0·1844	3·796	−1·010	0·2081	4·805	0	0·3974	17·61	12·81
4 × 10	0·1418	5·007	−1·286	0·1589	6·293	0	0·3023	23·16	16·86	0·4901	61·21	54·92	0·7317	136·67	130·37	1·5742	635·2	629·0	0·1681	4·163	−1·110	0·1896	5·273	0	0·3619	19·34	14·07
5 × 10	0·1322	5·371	−1·382	0·1481	6·752	0	0·2813	24·88	18·13	0·4555	65·86	59·11	0·6798	147·10	140·35	1·4620	684·0	677·2	0·1565	4·472	−1·175	0·1771	5·647	0	0·3366	20·80	15·15
6 × 10	0·1249	5·685	−1·464	0·1399	7·148	0	0·2653	26·39	19·24	0·4292	69·90	62·75	0·6402	156·20	149·05	1·3764	726·5	719·4	0·1477	4·739	−1·266	0·1665	6·005	0	0·3172	22·07	16·06
7 × 10	0·1191	5·961	−1·529	0·1335	7·491	0	0·2525	27·72	20·23	0·4082	73·49	66·00	0·6086	164·31	156·82	1·3080	764·5	757·0	0·1407	4·976	−1·331	0·1586	6·307	0	0·3017	23·20	16·89
8 × 10	0·1144	6·206	−1·594	0·1282	7·800	0	0·2421	28·91	21·11	0·3910	76·73	68·93	0·5826	171·64	163·84	1·2515	799·0	791·2	0·1349	5·190	−1·389	0·1520	6·579	0	0·2890	24·22	17·64
9 × 10	0·1105	6·425	−1·652	0·1238	8·078	0	0·2334	29·99	21·91	0·3765	79·68	71·60	0·5606	178·38	170·30	1·2033	830·7	822·6	0·1300	5·394	−1·442	0·1465	6·826	0	0·2783	25·15	18·33
1 × 10²	0·1072	6·623	−1·710	0·1200	8·333	0	0·2259	30·99	22·65	0·3640	82·42	74·08	0·5418	184·57	176·24	1·1628	860·0	851·7	0·1258	5·563	−1·491	0·1418	7·054	0	0·2690	26·02	18·96
2 × 10²	0·0886	8·016	−2·055	0·0993	10·072	0	0·1844	37·96	27·89	0·2934	102·25	92·18	0·4343	230·26	220·18	0·9260	1079·9	1069·8	0·1022	6·848	−1·838	0·1151	8·686	0	0·2159	32·42	23·74
3 × 10²	0·0801	8·661	−2·236	0·0901	11·096	0	0·1665	42·04	30·95	0·2614	114·77	103·67	0·3841	260·35	249·25	0·8135	1229·3	1218·2	0·0912	7·673	−2·054	0·1028	9·727	0	0·1918	36·50	26·78
4 × 10²	0·0750	9·463	−2·349	0·0847	11·812	0	0·1564	44·76	32·95	0·2427	123·61	111·80	0·3538	282·65	270·83	0·7435	1345·0	1333·2	0·0845	8·252	−2·233	0·0951	10·515	0	0·1774	39·47	28·95
5 × 10²	0·0715	9·930	−2·428	0·0809	12·358	0	0·1499	46·70	34·34	0·2303	130·26	117·91	0·3332	300·12	287·76	0·6946	1440·0	1427·3	0·0799	8·765	−2·349	0·0900	11·114	0	0·1676	41·75	30·64
6 × 10²	0·0689	10·309	−2·487	0·0782	12·796	0	0·1455	48·11	35·31	0·2217	135·32	122·52	0·3183	314·17	301·37	0·6581	1519·5	1506·7	0·0764	9·164	−2·436	0·0862	11·600	0	0·1606	43·57	31·97
7 × 10²	0·0668	10·629	−2·533	0·0760	13·161	0	0·1423	49·19	36·03	0·2154	139·28	126·11	0·3071	325·63	312·47	0·6295	1588·6	1575·4	0·0737	9·503	−2·506	0·0833	12·009	0	0·1554	45·05	33·04
8 × 10²	0·0651	10·905	−2·567	0·0742	13·472	0	0·1398	50·07	36·60	0·2107	142·38	128·91	0·2984	335·12	321·65	0·6064	1649·1	1635·6	0·0714	9·758	−2·561	0·0809	12·359	0	0·1512	46·28	33·92
9 × 10²	0·0637	11·148	−2·596	0·0728	13·744	0	0·1379	50·76	37·02	0·2071	144·86	131·11	0·2915	343·05	329·31	0·5874	1702·4	1688·7	0·0696	10·039	−2·609	0·0789	12·668	0	0·1479	47·31	34·65
1 × 10³	0·0625	11·364	−2·621	0·0715	13·984	0	0·1363	51·36	37·37	0·2042	146·91	132·93	0·2859	349·77	335·79	0·5714	1750·1	1736·1	0·0680	10·393	−2·648	0·0773	12·940	0	0·1452	48·20	35·26
2 × 10³	0·0556	12·781	−2·739	0·0644	15·521	0	0·1288	54·35	38·83	0·1922	156·09	140·57	0·2628	380·52	365·00	0·4910	2036·7	2021·1	0·0591	11·845	−2·884	0·0679	14·729	0	0·1324	52·87	38·14
3 × 10³	0·0522	13·602	−2·789	0·0610	16·391	0	0·1258	55·64	39·25	0·1887	158·98	142·59	0·2575	388·35	371·96	0·4638	2156·1	2139·7	0·0550	12·728	−2·975	0·0637	15·702	0	0·1279	54·71	39·01
4 × 10³	0·0501	14·180	−2·818	0·0588	16·998	0	0·1240	56·45	39·45	0·1873	160·17	143·17	0·2557	391·08	374·09	0·4527	2209·0	2192·0	0·0524	13·348	−3·014	0·0611	16·363	0	0·1255	55·77	39·40
5 × 10³	0·0485	14·630	−2·837	0·0573	17·467	0	0·1227	57·05	39·58	0·1866	160·77	143·30	0·2550	392·16	374·69	0·4478	2233·1	2215·7	0·0506	13·827	−3·037	0·0593	16·865	0	0·1239	56·48	39·62
6 × 10³	0·0473	14·998	−2·850	0·0560	17·848	0	0·1217	57·52	39·67	0·1862	161·12	143·27	0·2545	392·93	375·08	0·4455	2244·7	2226·8	0·0492	14·215	−3·052	0·0579	17·267	0	0·1228	57·02	39·76
7 × 10³	0·0464	15·308	−2·860	0·0550	18·169	0	0·1209	57·90	39·73	0·1859	161·36	143·21	0·2542	393·39	375·22	0·4444	2250·2	2232·1	0·0481	14·543	−3·061	0·0568	17·605	0	0·1213	57·45	39·85
8 × 10³	0·0456	15·560	−2·866	0·0542	18·447	0	0·1202	58·24	39·79	0·1856	161·64	143·19	0·2540	393·70	375·25	0·4436	2253·3	2234·8	0·0472	14·827	−3·069	0·0559	17·896	0	0·1211	57·81	39·92
9 × 10³	0·0449	15·620	−2·872	0·0535	18·692	0	0·1196	58·53	39·84	0·1853	161·90	143·21	0·2537	394·17	375·47	0·4434	2255·3	2236·6	0·0464	15·076	−3·074	0·0551	18·151	0	0·1204	58·12	39·97
1 × 10⁴	0·0443	16·034	−2·876	0·0529	18·911	0	0·1191	58·77	39·86	0·1851	162·07	143·16	0·2535	394·48	375·57	0·4431	2256·8	2237·9	0·0458	15·299	−3·079	0·0544	18·378	0	0·1199	58·39	40·01
2 × 10⁴	0·0406	17·471	−2·904	0·0491	20·375	0	0·1152	60·76	40·39	0·1828	164·11	143·74	0·2521	396·67	376·29	0·4420	2262·4	2242·1	0·0417	16·799	−3·068	0·0503	19·867	0	0·1165	60·07	40·23
3 × 10⁴	0·0388	18·308	−2·914	0·0471	21·222	0	0·1129	62·00	40·78	0·1812	165·56	144·34	0·2513	397·93	376·71	0·4418	2263·5	2242·2	0·0397	17·651	−3·077	0·0482	20·728	0	0·1148	60·99	40·27
4 × 10⁴	0·0376	18·903	−2·926	0·0458	21·829	0	0·1113	62·89	41·06	0·1800	166·67	144·84	0·2509	398·57	376·74	0·4418	2263·5	2241·6	0·0383	18·256	−3·083	0·0469	21·339	0	0·1136	61·63	40·29
5 × 10⁴	0·0367	19·372	−2·929	0·0448	22·302	0	0·1102	63·52	41·22	0·1792	167·41	145·11	0·2506	399·04	376·74	0·4418	2263·5	2241·2	0·0374	18·736	−3·087	0·0458	21·814	0	0·1127	62·13	40·30
6 × 10⁴	0·0359	19·755	−2·936	0·0441	22·691	0	0·1093	64·04	41·35	0·1786	167·97	145·28	0·2504	399·36	376·67	0·4417	2264·0	2241·3	0·0366	19·111	−3·090	0·0450	22·202	0	0·1120	62·53	40·33
7 × 10⁴	0·0354	20·079	−2·936	0·0435	23·015	0	0·1087	64·40	41·38	0·1781	168·44	145·43	0·2502	399·68	376·67	0·4417	2264·0	2241·0	0·0360	19·437	−3·093	0·0444	22·530	0	0·1113	62·87	40·34
8 × 10⁴	0·0349	20·361	−2·943	0·0429	23·305	0	0·1082	64·70	41·39	0·1778	168·73	145·42	0·2501	399·84	376·54	0·4417	2264·0	2240·7	0·0355	19·730	−3·096	0·0438	22·815	0	0·1108	63·16	40·34
9 × 10⁴	0·0344	20·616	−2·942	0·0425	23·557	0	0·1078	64·94	41·38	0·1776	168·92	145·36	0·2500	400·00	376·44	0·4417	2264·0	2240·4	0·0351	19·969	−3·098	0·0434	23·068	0	0·1104	63·41	40·34
1 × 10⁵	0·0341	20·839	−2·942	0·0421	23·731	0	0·1075	65·12	41·34	0·1774	169·11	145·33	0·2499	400·16	376·38	0·4416	2264·5	2240·7	0·0347	20·193	−3·101	0·0429	23·293	0	0·1100	63·64	40·35
2 × 10⁵	0·0317	22·412	−2·950	0·0394	25·361	0	0·1060	66·04	40·63	0·1694	177·10	151·73	0·2490	401·61	376·25	0·4414	2265·5	2240·2	0·0321	21·789	−3·028	0·0403	24·817	0			
3 × 10⁵	0·0305	23·302	−2·945	0·0381	26·247	0	0·1052	66·54	40·29	0·1692	177·31	151·06	0·2484	402·58	376·33	0·4412	2266·5	2240·3	0·0309	22·637	−3·052	0·0389	25·689	0			
4 × 10⁵	0·0297	23·938	−2·944	0·0372	26·882	0	0·1044	67·05	40·17	0·1691	177·41	150·53	0·2479	403·39	376·51	0·4411	2267·1	2240·2	0·0301	23·245	−3·066	0·0380	26·311	0			
5 × 10⁵	0·0291	24·441	−2·942	0·0365	27·382	0	0·1037	67·50	40·12	0·1689	177·62	150·24	0·2475	404·04	376·66	0·4410	2267·6	2240·2	0·0295	23·730	−3·076	0·0373	26·796	0			
6 × 10⁵	0·0286	24·651	−2·942	0·0360	27·793	0	0·1031	67·90	40·10	0·1688	177·73	149·93	0·2472	404·53	376·74	0·4409	2268·1	2240·3	0·0290	24·110	−3·083	0·0368	27·193	0			
7 × 10⁵	0·0282	25·204	−2·941	0·0355	28·145	0	0·1025	68·29	40·15	0·1687	177·83	149·69	0·2469	405·02	376·88	0·4408	2268·6	2240·5	0·0286	24·442	−3·088	0·0363	27·530	0			
8 × 10⁵	0·0278	25·503	−2·939	0·0352	28·441	0	0·1019	68·69	40·25	0·1685	178·04	149·60	0·2467	405·35	376·91	0·4407	2269·1	2240·7	0·0283	24·730	−3·091	0·0359	27·822	0			
9 × 10⁵	0·0276	25·762	−2·941	0·0348	28·703	0	0·1013	69·10	40·40	0·1684	178·15	149·44	0·2466	405·52	376·81	0·4407	2269·1	2240·4	0·0280	24·986	−3·095	0·0356	28·080	0			
1 × 10⁶	0·0273	25·998	−2·937	0·0346	28·935	0	0·1008	69·44	40·51	0·1683	178·25	149·32	0·2464	405·84	376·91	0·4407	2269·1	2240·2	0·0278	25·214	−3·096	0·0353	28·312	0			

At large values of x^+ *and* large values of N_{Pr}, the function has the asymptotic form:*

$$S_T \rightarrow \frac{\sin(\pi/4)}{(\pi/4)} \cdot \frac{K^{5/4}}{(E4!)^{1/4}} \cdot N_{Pr}^{1/4}. \tag{23}$$

The first of these asymptotes results from substituting 1 for ϵ_u^+ and $1/N_{Pr}$ for ϵ_h^+ in equation (1) as shown in the Appendix. It is the solution of Leveque [5], Owen and Ormerod [6] and Lighthill [7]. The second asymptote is obtained by neglecting the left-hand side of (1), substituting 1 for ϵ_u^+ and taking only the first two terms of the expansion for ϵ_h^+. It is easily shown that these substitutions are appropriate. With $K = 0{\cdot}4$ and $E = 9{\cdot}025$, the right-hand side of equation (23) is equal to $0{\cdot}0746\ N_{Pr}^{1/4}$; with $K = 0{\cdot}407$ and $E = 10{\cdot}09$, its value is only slightly different. Accordingly ordinates equal to $0{\cdot}0746\ N_{Pr}^{1/4}$ have been marked as a scale of N_{Pr} on the right-hand margin of Fig. 1.

The S_q function computed in reference [4] is also plotted in Fig. 1 (broken lines). At low values of x^+, the function has the asymptotic form:†

$$S_q \rightarrow 0{\cdot}651\,(x^+/N_{Pr})^{1/3}. \tag{24}$$

The derivation of this expression is given in the Appendix. At large x^+ *and* large N_{Pr}, the S_q function tends to the same asymptote as the S_T function namely:

$$S_q \rightarrow \frac{\sin(\pi/4)}{(\pi/4)} \cdot \frac{K^{5/4}}{(E4!)^{1/4}} N_{Pr}^{1/4}. \tag{25}$$

Figure 2 is based on values of ξ and u^+, which have been tabulated by the authors of references [3] and [4]. Examination of equation (7) in conjunction with (20) and (21) shows that, at large u^+ and N_{Pr}, the $\xi \sim u^+$ relation takes the asymptotic form:

$$\xi \rightarrow u^+ + \frac{\pi/4}{\sin(\pi/4)} \cdot \frac{(E4!)^{1/4}}{K^{5/4}} (N_{Pr} - 1)\, N_{Pr}^{-1/4}. \tag{26}$$

With $K = 0{\cdot}4$ and $E = 9{\cdot}025$, this becomes:

$$\xi \rightarrow u^+ + 13{\cdot}4\,(N_{Pr} - 1)\, N_{Pr}^{-1/4}. \tag{26a}$$

At low u^+, on the other hand, it is equally obvious that ξ is equal to $N_{Pr}\, u^+$. Figure 2 also

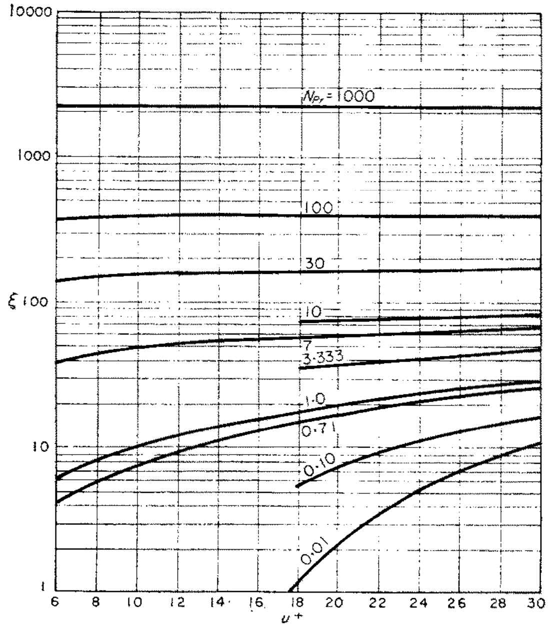

FIG. 2. Values of $\xi(u^+, N_{Pr})$ from references [3], [4] and [8]. When generalized to $N_{Pr,t} \neq 1$, the ordinate becomes $\xi/N_{Pr,t}$ and the parameter $N_{Pr}/N_{Pr,t}$.

contains some curves based on tables of ξ computed by Mills [8]. It should be noted that, as is shown in the next section, the symbol ξ has the same significance as the symbol t^+ used by Deissler [12] and other authors for the dimensionless temperature in Couette-flow analyses of heat transfer.

Table 1 contains values of $(\xi - u^+)$ deduced from the tables presented in references [3], [4] and [8]. The constancy of $(\xi - u^+)$ at large u^+ is evident. Also included for comparison are values of $13{\cdot}4\,(N_{Pr} - 1)\, N_{Pr}^{-1/4}$.

2.4 *Regularities in the existing solutions at moderate and large x^+*

Table 2 contains the values of S_T and S_q which are reported in references [3] and [4] together with corresponding values of N_{Pr}/S, i.e.

* The Gardner–Kestin [3] solutions show significant deviations from this asymptote, even at large Prandtl number, perhaps because of the accumulation of rounding errors in the computations.

† The Smith–Shah [4] solutions show significant deviations from this asymptote, presumably because the integration intervals were excessively large at low x^+.

2X

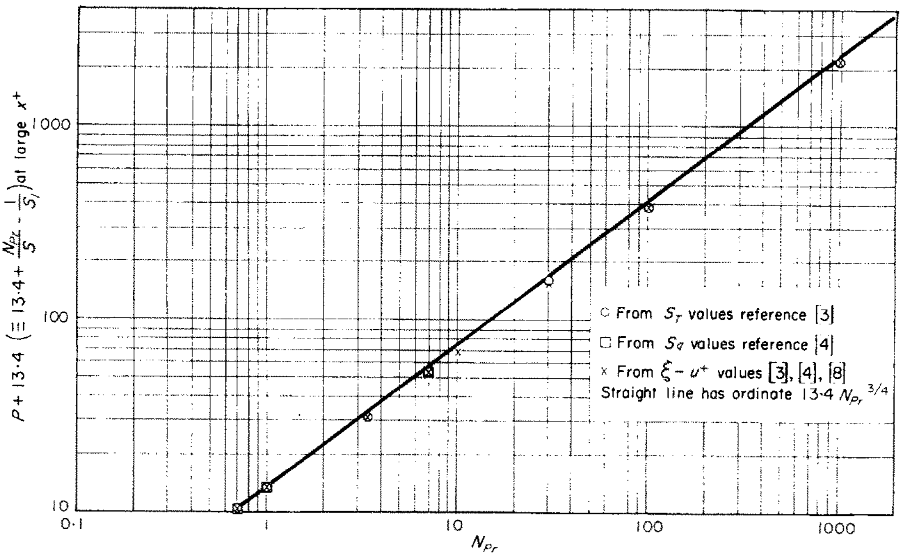

FIG. 3. The P-function deduced from exact solutions. When generalized for $N_{Pr,t} \neq 1$, N_{Pr} must be replaced by $N_{Pr}/N_{Pr,t}$.

$(c_f/2)^{1/2}/N_{St}$, and $[N_{Pr}/S - 1/S_1]$ where S_1 refers to the corresponding value for $N_{Pr} = 1$ and the same x^+. (N.B. Actually, $x^+/N_{Pr,t}$ appears in Table 2 rather than x^+, and $N_{Pr}/N_{Pr,t}$ in place of N_{Pr}. The presence of $N_{Pr,t}$ will be ignored in the present discussion, that is to say that its value will be assumed to be unity. The reason for including $N_{Pr,t}$ will become apparent in section 3.3.)

Inspection of the column for $[(c_f/2)^{1/2}/N_{St} - 1/S_1]$ shows that this quantity exhibits only small variations for x^+ values greater than or equal to 10^4. This behaviour may be expressed as:

$$x^+ \geqslant 10^4: \ N_{Pr}/S \approx 1/S_1 + P \qquad (27)$$

where P is a function of Prandtl number, and is slightly dependent on whether the S_T or the S_q function is in question. The variations are probably as much due to minor inaccuracies in the tables of references [3] and [4] as to any other cause.

Figure 3 shows a plot of P, or rather $13{\cdot}4 + P$, versus N_{Pr}. The values of P are the values which Table 2 shows are taken by $N_{Pr}/S - 1/S_1$ at large x^+. Circles represent deductions from S_T values, squares represent deductions from S_q values. Also shown, as crosses, are some values taken by $\xi - u^+ + 13{\cdot}4$ at large u^+, deduced from Table 1; evidently they lie close to the P values.

The extent to which equation (27) is obeyed, and the ease of interpolation in Fig. 3, suggest an obvious and easy way of extending the existing tables of S_T and S_q. Before exploring this possibility however, it is fruitful to examine the reasons for the great regularity which the S_T and S_q functions have been seen to possess.

2.5 *Explanation of equation* (27)

The following argument explains why the S_T and S_q functions obey equation (27) at moderate and large x^+:

(i) Consider the function A_q (x^+) which is related to the wall temperature downstream of a line sink of heat, of strength $\dot{q}'$ (e.g. in Btu/ft h), in an otherwise adiabatic wall, by:

$$T_G - T_S = (\dot{q}'/c\mu)\, A_q. \qquad (28)$$

This function can be obtained by solving the partial differential equation (1) subject to the conditions:

$$\left.\begin{aligned} &\left.\begin{aligned} &x^+ = 0,\ u^+ \text{ (or } \xi) > 0 \\ &\text{all } x^+,\quad u^+ \text{ (or } \xi) \to \infty \end{aligned}\right\}: \ T = T_G \\ &x^+ > 0,\ u^+ \text{ (or } \xi) = 0: (\partial T/\partial \xi) = 0 \\ &x^+ > 0: \int_0^\infty (T - T_G)\, c\, u^+\, \epsilon_u^+\, du^+ = \dot{q}'/c\mu \end{aligned}\right\} \qquad (29)$$

Familiarity with the corresponding problem of heat-conduction theory, namely that of the transient temperature distribution in a medium subsequent to the release of an instantaneous heat source, makes it obvious that successive temperature profiles will have the qualitative features of those sketched in Fig. 4: specifically, the profile will become broader and lower as x^+ increases, and will exhibit a maximum at the wall.

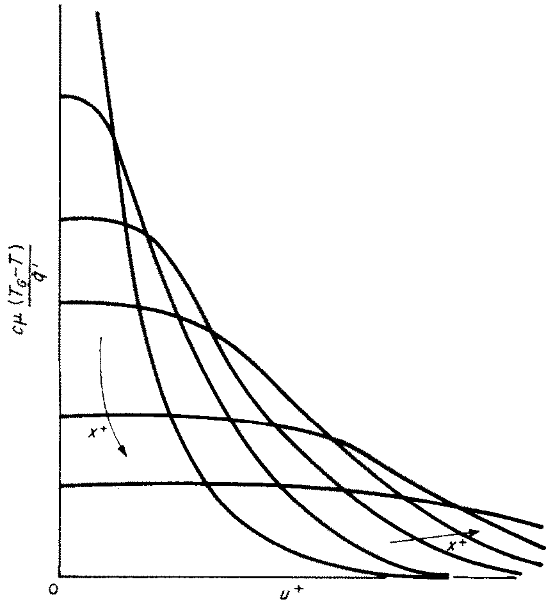

FIG. 4. Sketch of temperature distributions at successive sections through a turbulent boundary layer downstream of a line heat sink.

(ii) Now the (laminar) Prandtl number can only influence the heat-transfer process in the immediate vicinity of the wall, as may be seen from consideration of equation (21) for example. Yet, as Fig. 4 shows, the temperature gradients in this region are negligible except immediately downstream of the heat sink (small x^+). Since the Prandtl number cannot exert its influence in the absence of a temperature gradient, the A_q *function must be independent of Prandtl number at moderate and large* x^+.

(iii) The S_q function can be generated from the A_q function by regarding a continuous heat flux $\dot{q}''$ as made up of a series of line sinks.* We see that the wall temperature must be:

$$T_S - T_G = \int_0^x \frac{A_q}{c\mu}\dot{q}''\,\mathrm{d}x = \int_0^{x^+} A_q \frac{\dot{q}''}{c(\tau_S\rho)^{1/2}}\,\mathrm{d}x^+ \qquad (30)$$

and so, since $\dot{q}''/[c(\tau_S\rho)^{1/2}] = \text{const.}$ by hypothesis and S_q is defined by equation (19), we have:

$$\frac{N_{Pr}}{S_q} = \int A_q\,\mathrm{d}x^+$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x^+}\left(\frac{N_{Pr}}{S_q}\right) = A_q. \qquad (31)$$

(iv) But, as explained in (ii), A_q must be independent of N_{Pr} for moderate and large x^+. It follows that the Prandtl number can only influence the value of N_{Pr}/S_q by way of the integration constant in this x^+ range. In other words, N_{Pr}/S_q must be equal to a function of x^+ plus a function of Prandtl number. Obviously $(S_{q,1})^{-1}$ provides a suitable version of the function of x^+.

(v) As inspection of Fig. 1 shows, S_T and S_q are nearly equal in the Prandtl number range in question; moreover they have the same asymptote at high x^+ and N_{Pr} [equations (23) and (25)]. It is, therefore, to be expected that N_{Pr}/S_T differs from $1/S_{T,1}$ at the same x^+ by a quantity which depends on N_{Pr} alone. This is what it was desired to prove.

Equation (27) can also be interpreted in another way: N_{Pr}/S is equal, according to equation (19), to $(T_G - T_S)c(\tau_S\rho)^{1/2}/\dot{q}''$; it thus has the significance of the non-dimensional "resistance" to heat transfer (temperature difference per unit heat flux) of the whole boundary layer. $1/S_1$ is therefore the "resistance" which the boundary layer would have if the fluid Prandtl number were unity. If P is *defined* as $N_{Pr}/S - 1/S_1$, it can be called the "extra resistance" associated with the fact that the Prandtl number differs from unity. Equation (27) therefore implies that this "extra resistance" is dependent on Prandtl number alone at moderate and large values of x^+.

The same interpretation aids the understanding of why the asymptotic values of $\xi - u^+$ are closely equal to the asymptotic values of P,

* This procedure for deriving S_q is used in the Appendix.

as Fig. 3 clearly shows. For, in the analysis of heat flow through a boundary layer having no temperature variation in the x^+ direction, i.e. in a Couette-flow analysis, we should calculate:

$$t^+ \equiv c\frac{(T - T_S)(\tau_S \rho)^{1/2}}{\dot{q}''} = \int_0^{u^+} \frac{\epsilon_u^+}{\epsilon_h^+} du^+ =$$

$$= u^+ + \int_0^{u^+} \left(\frac{\epsilon_u^+}{\epsilon_h^+} - 1\right) du^+. \quad (32a)$$

Now comparison of (32a) with (7) shows that ξ is exactly equivalent to t^+, the non-dimensional temperature appearing in the Couette-flow analysis; t^+ itself, when described in the terms of the last paragraph, can itself be regarded as the "resistance to heat transfer" of the layer of "thickness" u^+. So the "extra resistance" due to the fact that the Prandtl number is not unity is given by the quadrature appearing in equation (32)a, and is also equal to $\xi - u^+$. It is no surprise that $\xi - u^+$ is a function of Prandtl number alone for moderate and large u^+, for it is only in the low u^+ region that the integrand of the quadrature is finite. We have already seen, in equations (26) and (26a), particular forms of the $\xi - u^+$ asymptote, calculated by inserting (20) and (21) in (32a).

Summarizing, we may conclude that exact solutions of the partial differential equation of heat transfer across a turbulent boundary layer have shown that, at moderate and large values of x^+, the effects of the Prandtl number of the fluid are confined to the value of the effective "resistance" to heat transfer of a thin layer adjacent the wall. Convective (i.e. $\partial/\partial x^+$) terms only need to be considered in regions where the fluid Prandtl number is without influence. This conclusion may be regarded as obvious; nevertheless it could not have been drawn with such complete certainty had the authors of references [2], [3] and [4] not computed their exact solutions of the partial differential equation. The conclusion can be regarded as a limited rehabilitation of the Couette-flow analysis.

2.6 *Approximate expressions for the P-function*

The significance of the $P(N_{Pr})$ function may become clearer if the reader is reminded that a Couette-flow analysis of the kind originated by Prandtl [9] and Taylor [10], would lead us to write:

$$u^+ \geqslant u_1^+ : \xi - u^+ = (N_{Pr} - 1)\, u_1^+ \quad (32b)$$

where u_1^+ is a constant, variously taken to be 5·68 or 8·9, which measures the non-dimensional thickness of the laminar sub-layer. Other authors including von Kármán [18], Rannie [19] and Deissler [12] have suggested other forms of the P function, but it would be inappropriate to review these here. We merely take equation (32b) as a reminder of the convenience of having an analytical expression for the P function.

There are three aspects of the P function which require attention at the present juncture:

(i) Approximately, the P values displayed on Fig. 4 can be represented by the relation:

$$P = 13{\cdot}4\,(N_{Pr}^{3/4} - 1). \quad (33)$$

The coefficient, 13·4, is, as already stated, the value of the expression $(\pi/4)(E4!)^{1/4}/[5^{5/4}\sin(\pi/4)]$ appearing in equations (23), (25) and (26), for $K = 0{\cdot}4$ and $E = 9{\cdot}025$; equation (33) is therefore certainly correct at both infinite and unity Prandtl number. The equation is shown as a straight line in Fig. 3.

The fact that the points obtained from Table 2 tend to lie below the line is almost certainly due to inaccuracies in the Table 2 values. The values of P at low Prandtl number contained in Mills' [8] report have not been included on the graph, but they lie appreciably below the straight line.

(ii) It would no doubt be possible to find an analytical relation between P and N_{Pr} which was in even closer agreement with the points marked on Fig. 3 than is equation (33). However, it must not be forgotten that, though the ϵ_u^+ function of equation (20) is known to agree well with the experimental velocity distribution data [11], the ϵ_h^+ function of equation (21) is little more than a guess; no attempt whatever has been made to choose this so as to fit experimental measurements. In this connexion it is relevant to mention that Deissler [12] has made use of an ϵ_h^+ expression which, like that of equation (21), increases in proportion to $(u^+)^4$ near the wall. Deissler's expression contains however, a variable coefficient, the value of which was chosen so as to fit experimental data for heat and mass transfer at high Prandtl or

Table 3. Comparison of exact and approximate values of S_T and S_q. Moderate values of x^+

Isothermal wall

N_{Pr}	0·71		7·0		30		100		1000	
x^+	S_T from Table 2	S_T from equation (38)	S_T from Table 2	S_T from equation (38)	S_T from Table 2	S_T from equation (38)	S_T from Table 2	S_T from equation (38)	S_T from Table 2	S_T from equation (38)
10^4	0·04428	0·04456	0·1191	0·1193	0·1851	0·1852	0·2535	0·2536	0·4431	0·4431
10^5	0·03407	0·03325	0·1075	0·1067	0·1774	0·1769	0·2499	0·2496	0·4416	0·4415
10^6	0·02731	0·02497	0·1008	0·0974	0·1683	0·1660	0·2464	0·2449	0·4407	0·4402

Prescribed heat flux

N_{Pr}	0·7		7·0	
x^+	S_q from Table 2	S_q from equation (39)	S_q from Table 2	S_q from equation (39)
10^4	0·04575	0·04546	0·1199	0·1197
10^5	0·03467	0·03371	0·1100	0·1090
10^6	0·02776	0·02525		

Table 4. Comparison of exact and approximate values of S_T and S_q. Large x^+ range

Isothermal wall

N_{Pr}	0·71		7·0		30		100		1000	
x^+	S_T from Table 2	S_T from equation (41)	S_T from Table 2	S_T from equation (41)	S_T from Table 2	S_T from equation (41)	S_T from Table 2	S_T from equation (41)	S_T from Table 2	S_T from equation (41)
10	0·2228	0·2282	0·4779	0·4787	0·7763	0·7770	1·1598	1·1603	2·4984	2·4993
10^2	0·1072	0·1141	0·2259	0·2274	0·3640	0·3655	0·5418	0·5433	1·1628	1·1648
10^3	0·0625	0·0667	0·1363	0·1330	0·2042	0·2027	0·2859	0·2889	0·5714	0·5823
10^4	0·0443	0·0457	0·1191	0·1119	0·1851	0·1711	0·2535	0·2365	0·4431	0·4314
10^5	0·0341	0·0335	0·1075	0·1022	0·1774	0·1643	0·2499	0·2302	0·4416	0·4184
10^6	0·0273	0·0251	0·1008	0·0925	0·1683	0·1581	0·2464	0·2264	0·4407	0·4165

Prescribed heat flux

N_{Pr}	0·7		7·0	
x^+	S_q from Table 2	S_q from equation (42)	S_q from Table 2	S_q from equation (42)
10	0·2667	0·2718	0·5729	0·5784
10^2	0·1258	0·1317	0·2690	0·2716
10^3	0·0680	0·0718	0·1452	0·1452
10^4	0·0458	0·0470	0·1199	0·1135
10^5	0·0347	0·0339	0·1100	0·1029
10^6	0·0278	0·0253		

Schmidt numbers. If that value were adopted here, it would be reasonable to replace equation (33) by:

$$P = 8{\cdot}95\,(N_{Pr}^{3/4} - 1). \tag{34}$$

It may be expected that equation (34) would lead to more correct heat-transfer predictions than the values of P implicit in references [3] and [4].

(iii) In any case, however, there is no necessity to *guess* the nature of the P function, or to force it to fit any particular analytical expression; for the function can be derived directly from inspection of the prolific heat-transfer data which are available. This has been done by one of the author's co-workers; the result will be the subject of a separate report. In what follows, we merely assume that the function $P(N_{Pr})$ exists.

2.7 *Approximate expressions for the S_T and S_q functions for moderate values of x^+*

Relation between x^+ and $N_{Pr,x}$ for a flat plate. Since "length Reynolds number" is a more familiar concept than x^+, it will be helpful to establish the relation between these quantities for the particular case of flow of uniform stream velocity u_G. It will be sufficiently accurate to start from the approximate law for local drag:

$$\frac{c_f}{2} = 0{\cdot}0296\left(\frac{u_G x \rho}{\mu}\right)^{-1/5} \equiv 0{\cdot}0296\,(N_{Re,x})^{-1/5}. \tag{35}$$

The definition of x^+ (2) now permits the easy derivation of the following relation:

$$x^+ = 0{\cdot}191\,(N_{Re,x})^{0\cdot 9}. \tag{36}$$

Thus the common $N_{Re,x}$ range: 2×10^5 to 5×10^7 corresponds roughly to the x^+ range: 10^4 to 10^6. We shall now derive approximate expressions for the quantities S_T and S_q which are valid in this range.

The S_T function. It is shown in the Appendix that, when ϵ_u^+ has the form which corresponds to the well-known "seventh-power velocity profile", and the total Prandtl number is unity,

$$S_{T,1} = 0{\cdot}1479\,(x^+)^{-1/9}. \tag{37}$$

Treating equation (27) as exact, we can derive a corresponding expression for S_T. It is:

$$S_T = N_{Pr}\,[6{\cdot}76\,(x^+)^{1/9} + P]^{-1}. \tag{38}$$

This expression, with P inserted from equation (33), will be found to agree closely with the values calculated by Gardner and Kestin [3]; the comparison is made in Table 3. With P inserted from equation (34), better agreement with experimental data might be expected. It should be noted that the "seventh-power profile" is a good approximation to the real velocity profile in the Reynolds-number and x^+ ranges in question.

The S_q function. It is shown in the Appendix that the "seventh-power profile" assumption also implies:

$$S_{q,1} = 0{\cdot}1509\,(x^+)^{-1/9}. \tag{39}$$

It may therefore be expected that, for $10^4 \leqslant x^+ \leqslant 10^6$, the following relation should yield S_q values in good agreement with those of Smith and Shah [4]:

$$S_q = N_{Pr}\,[6{\cdot}64\,(x^+)^{1/9} + P]^{-1} \tag{40}$$

with P inserted from equation (33). Table 3 again shows that the expectations are realized. The use of equation (34) for the P function might, once again, yield better agreement with experiment.

2.8 *Approximate expressions for the S_T and S_q functions for moderate and small values of x^+*

It may be desired to use approximate formulae for S_T and S_q which are also valid at arbitrarily small values of x^+, i.e. at those for which the thermal boundary layer is confined almost wholly within the laminar sub-layer. Since the asymptotic relations (22) and (24) are available to act as a guide, it is fairly easy to derive formulae which fit the exact solutions both at low x^+ and at values between 10^4 and 10^6, and which have a good chance of fitting at intermediate values also. The following relations may be found to involve a reasonable compromise between accuracy and complexity.

$$S_T = \left\{\left[\frac{N_{Pr}}{6{\cdot}76\,(x^+)^{1/9} + P}\right]^4 + \left[0{\cdot}53835\left(\frac{x^+}{N_{Pr}}\right)^{-1/3}\right]^4\right\}^{1/4} \tag{41}$$

and

$$S_q = \left\{\left[\frac{N_{Pr}}{6\cdot64\,(x^+)^{1/9} + P}\right]^4 + \right.$$
$$\left. + \left[0\cdot651\left(\frac{x^+}{N_{Pr}}\right)^{-1/3}\right]^4\right\}^{1/4}. \quad (42)$$

These equations are readily seen to reduce respectively to equations (22) and (24) at low x^+, and to (38) and (40) at large x^+. If P is inserted from equation (33), they can be expected to yield results in good agreement with those of references [3] and [4]. This last expectation may be checked by reference to Table 4, in which comparison is made between the exact and the approximate values.

Equations (41) and (42) may of course be used for filling in the as-yet-uncharted regions of Fig. 1. Indeed, in view of the uncertainty about the exact nature of the ϵ_h^+ function and the corresponding $P\,(N_{Pr})$ function, the equations may be thought to render unnecessary any further computations of the kind presented in [3] and [4] until the said uncertainty has been removed by further examination of experimental data.

3. GENERALIZATION OF THE THEORY FOR NON-UNITY PRANDTL NUMBER IN THE TURBULENT REGION

3.1 *The necessity for generalization*

One of the main assumptions of the Couette-flow analysis has been shown to be well-justified, namely the assumption that the laminar-sub-layer region acts solely as a "resistance" to heat transfer. It is therefore profitable to recall that the Couette-flow analysis for heat transfer from an isothermal flat plate is in marked disagreement with experiment, when the "total" Prandtl number in the turbulent region is taken as unity. For such a situation, with $N_{St,1}$ placed equal to $c_f/2$ as is implied by the turbulent-Prandtl-number assumption, equation (27) would lead to:

$$\frac{c_f/2}{N_{St}} = 1 + (c_f/2)^{1/2}P. \quad (43)$$

The expression on the left-hand side is the so-called Reynolds analogy factor; on the right-hand side, P might be substituted from equations (33) or (34). In the former case, for a Prandtl number of 0·7, P has the value: $-3{\cdot}04$.

Now the presence of $(c_f/2)^{1/2}$, which varies with the Reynolds number of the plate, implies that the Reynolds analogy factor is appreciably influenced by $N_{Re,x}$. Thus, if $c_f/2$ is calculated from equation (35), $(c_f/2)N_{St}$ equals 0·816 at $N_{Re,x} = 10^5$ and 0·927 at $N_{Re,x} = 10^7$. Experimental data do not reveal nearly so strong an effect of Reynolds number.

Similar conclusions may be drawn from inspection of recovery-factor data, which again fail to reveal the strong Reynolds number effect which would exist if the "total" Prandtl number in the turbulent region were unity.

Whereas in the past it has been possible to ascribe the discrepancies between prediction and experiment to the fact that the Couette-flow analysis neglected the differential coefficients with respect to x, the findings of section 2 of the present paper show that this explanation is untenable. It is therefore necessary to examine whether the solutions of the partial differential equation published by Gardner and Kestin [3] and by Smith and Shah [4] can be generalized to non-unity "total" Prandtl number, or whether they must be discarded and replaced by solutions based on a new ϵ_h^+ function.

3.2 *Experimental evidence for the value of the turbulent Prandtl number*

Kestin and Richardson [13], in their review of heat transfer across turbulent incompressible boundary layers, have collected data for $N_{Pr,t}$ deduced by several authors from measurements in turbulent pipe flows. Figure 5 (Fig. 11 of [13]) illustrates their findings. As may be seen, the various curves show marked disagreement with each other, but nevertheless strongly suggest that, if a constant value is to be assumed for $N_{Pr,t}$, 0·8 would be more reasonable than 1·0.

Van Driest [14] and Spence [15] have developed theories for the Reynolds analogy and recovery factors of a flat plate in turbulent flow which account almost completely for the convective $(\partial/\partial x)$ terms in the partial differential equation and which allow $N_{Pr,t}$ to be a constant, different from unity. Comparison of the results of these theories, which differ more in appearance than

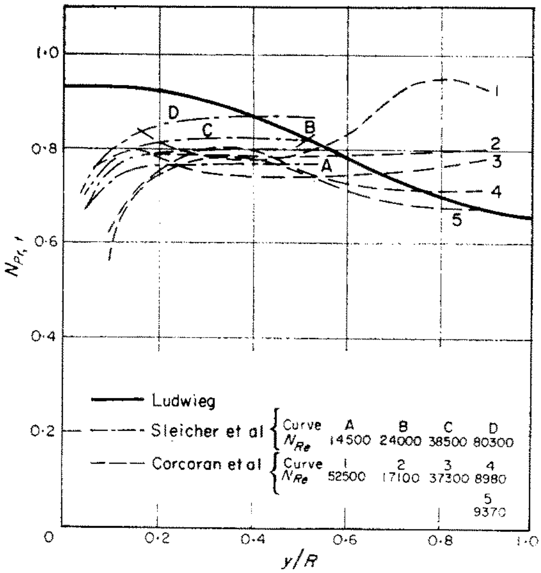

FIG. 5. Values of $N_{Pr,t}$ vs. non-dimensional distance from wall in turbulent pipe flow, collected by Kestin and Richardson [13].

in essence, with experimental data for air suggest that $N_{Pr,t}$ is about 0·86.

Values of $N_{Pr,t}$ may also be deduced from measurements of the temperature profile in fully developed turbulent pipe flow. Such measurements have recently been reported by Johnk and Hanratty [16] who found that, for values of y^+ from thirty to two or three hundred, the temperature profiles could be represented by:

$$t^+ = 3{\cdot}3 + 5{\cdot}1 \log_{10} y^+. \tag{44}$$

Here y^+ is of course $y(\tau_S \rho)^{1/2}/\mu$, where y is distance from the wall, and t^+ is $(T - T_S)\, c(\tau_S\rho)^{1/2}/\dot{q}''$. It is well known [17] that, in precisely this y^+ range, the velocity profile can be expressed by:

$$u^+ = 5{\cdot}5 + 5{\cdot}75 \log_{10} y^+. \tag{45}$$

Elimination of y^+ between (44) and (45) leads to:

$$t^+ = 0{\cdot}887\, u^+ - 1{\cdot}58 \tag{46}$$

We shall now show that the coefficient of u^+ can be identified with $N_{Pr,t}$.

By differentiation of (46) we obtain:

$$\frac{dt^+}{dy} = 0{\cdot}887 \frac{du^+}{dy}. \tag{47}$$

When t^+ and u^+ are re-written in terms of T, u and other variables, this equation becomes:

$$c \cdot \frac{1}{\dot{q}''}\frac{dT}{dy} = \frac{0{\cdot}887}{\tau_S}\frac{du}{dy}. \tag{48}$$

If the local heat flux is equal to the heat flux at the wall, $\dot{q}''$, we may write:

$$c \cdot \frac{1}{k_t} = \frac{0{\cdot}887}{\mu_t}$$

i.e.

$$N_{Pr,t} = 0{\cdot}887. \tag{49}$$

This value of the turbulent Prandtl number is in good agreement with the value of 0·86 deduced in references [14] and [15].

We can also learn something about the resistance of the laminar sub-layer in the experiments of reference [16] from the fact that the additive constant in equation (46) is $-1{\cdot}58$.

Let us suppose, generalizing equations (20) and (21), that:

$$\epsilon_u^+ = 1 + \phi\,(u^+) \tag{50a}$$

and:

$$\epsilon_h^+ = (1/N_{Pr}) + (1/N_{Pr,t})\,\phi\,(u^+) \tag{50b}$$

where N_{Pr} of course refers to the *laminar* Prandtl number and $N_{Pr,t}$ is a constant. In a Couette-flow analysis, differential coefficients with respect to x are neglected. Then equation (1) becomes an ordinary differential equation, with solution:

$$t^+ = N_{Pr,t}\, u^+ + N_{Pr,t}\left(\frac{N_{Pr}}{N_{Pr,t}} - 1\right) \int_0^{u^+} \frac{1}{1 + (N_{Pr,t}/N_{Pr})\,\phi}\, du^+. \tag{51}$$

The second term of equation (51) can be identified as the extra resistance of the laminar sub-layer, associated with the fact that N_{Pr} is not equal to $N_{Pr,t}$; let us call this:

$$N_{Pr,t}\, P\,(N_{Pr}/N_{Pr,t}).$$

Generalizing equations (33) and (34) we may guess that P is equal to a constant times $[(N_{Pr}/N_{Pr,t})^{3/4} - 1]$; then, equating $N_{Pr,t}\,P$ to the $-1{\cdot}58$ appearing in equation (46), with $N_{Pr,t} = 0{\cdot}887$ and $N_{Pr} = 0{\cdot}71$, we obtain:

$$P = 11{\cdot}57\,[(N_{Pr}/N_{Pr,t})^{3/4} - 1]. \tag{52}$$

Thus the data of Johnk and Hanratty yield a coefficient rather nearer to that appropriate to [3], [4] and [8] than to that deduced from Deissler's recommendation.

Of course this brief review does not finally settle the questions of whether $N_{Pr,t}$ can be taken as a constant, independent of N_{Pr} and other influences, of what is the best value of the constant, or of what is the relation between P and $N_{Pr}/N_{Pr,t}$. Nevertheless it suggests that a theory based on equations (49) and (52) should give fairly good predictions of heat transfer for air. It further encourages the development of a complete heat-transfer theory based on a constant value of $N_{Pr,t}$; this development now follows.

3.3 *Generalization of the solutions of* [3] *and* [4]

We shall now show that the solutions displayed in Fig. 1 are still valid for $N_{Pr,t} = \text{const} \neq 1$, provided that appropriate adjustments are made to the expressions for abscissa and parameter. Equation (1) may be written, on introduction of (50a) and (50b), as:

$$\frac{\partial T}{\partial x^+} = \frac{1}{u^+ (1 + \phi)} \frac{\partial}{\partial u^+} \times \left[\frac{(1/N_{Pr}) + (1/N_{Pr,t}) \phi}{1 + \phi} \cdot \frac{\partial T}{\partial u^+}\right]. \tag{53}$$

This equation can be re-arranged as:

$$\frac{\partial T}{\partial (x^+/N_{Pr,t})} = \frac{1}{u^+ (1 + \phi)} \frac{\partial}{\partial u^+} \left\{\frac{(N_{Pr,t}/N_{Pr}) + \phi}{1 + \phi} \frac{\partial T}{\partial u^+}\right\}. \tag{54}$$

Now Gardner and Kestin [3] and Smith and Shah [4] solved the equation:

$$\frac{\partial T}{\partial x^+} = \frac{1}{u^+ (1 + \phi)} \frac{\partial}{\partial u^+} \left\{\frac{(1/N_{Pr}) + \phi}{1 + \phi} \frac{\partial T}{\partial u^+}\right\} \tag{55}$$

with, of course, a particular expression for ϕ, namely: $(K/E)\,[\exp\,(Ku^+) - 1 - Ku^+ - (Ku^+)^2/2! - (Ku^+)^3/3!]$, which has been shown in [11] to fit experimental velocity distributions rather well. It follows that the solutions contained in [3] and [4] are still valid for $N_{Pr,t} \neq 1$, provided that they are expressed in the form $S\,(x^+/N_{Pr,t},\ N_{Pr}/N_{Pr,t})$ instead of $S\,(x^+,\ N_{Pr})$.

It must be emphasized that this generalization of the solutions of [3] and [4] is only strictly valid when the thermal boundary layer starts from a line on the wall at which the velocity boundary layer already has an appreciable thickness. When this condition is not satisfied, and when $N_{Pr,t}$ is less than unity, account needs to be taken of the fact that the shear stress within the thermal boundary layer is not uniform, for example by the method used in Spence's [15] theory. This matter will not be discussed further here.

3.4 *Generalization of the approximate expressions for S_T and S_q*

One way of generalizing equation (38) to the case in which $N_{Pr,t}$ is not equal to unity, is to note that equation (A.21) of the Appendix shows the thermal resistance of the turbulent region to be approximately:

$$\frac{N_{Pr,t}}{S_{T,1}} = 6{\cdot}76 \left(\frac{x^+}{N_{Pr,t}}\right)^{1/9} N_{Pr,t}. \tag{56}$$

Since the extra thermal resistance afforded by the laminar sub-layer is $N_{Pr,t}\,P\,(N_{Pr}/N_{Pr,t})$, the total resistance to heat transfer is:

$$\frac{N_{Pr}}{S_T} = \frac{N_{Pr}}{S_{T,1}} + N_{Pr,t}\,P. \tag{57}$$

We therefore conclude, making use of equation (56):

$$\frac{N_{St}}{(c_f/2)^{1/2}} = \frac{S_T}{N_{Pr}} = \frac{1}{N_{Pr,t}} \left[6{\cdot}76 \left(\frac{x^+}{N_{Pr,t}}\right)^{1/9} + P\right]^{-1}. \tag{58}$$

A similar argument applied to the case of uniform $\dot{q}''/(\tau_S)^{1/2}$ leads to:

$$\frac{N_{St}}{(c_f/2)^{1/2}} = \frac{S_q}{N_{Pr}} = \frac{1}{N_{Pr,t}} \left[6{\cdot}64 \left(\frac{x^+}{N_{Pr,t}}\right)^{1/9} + P\right]^{-1}. \tag{59}$$

Equations (58) and (59), with P inserted from equation (52) and $N_{Pr,t}$ placed equal to 0·887, probably represent the best simple recommendations which can currently be made for calculating heat transfer through incompressible boundary layers in the x^+ range from 10^4 to 10^6. For the isothermal flat plate in air, if we ignore the fact that the equal lengths of the thermal and velocity boundary layers renders the applicability of the theory questionable, we obtain, from equations (58), (35) and (36): $(c_f/2)/N_{St} = 0{\cdot}817$ for $x^+ = 10^4$ and $(c_f/2)/N_{St} = 0{\cdot}835$ for $x^+ = 10^6$. These values exhibit a small effect of Reynolds number and a mean value that is in excellent agreement with the experimental value which is usually taken to equal 0·825.

It is possible in the same manner to generalize equations (41) and (42) and so to obtain expressions for S_T and S_q which are valid over a wider range of x^+ than those of (58) and (59). The expressions are:

$$S_T = \left\{\left[\frac{N_{Pr}/N_{Pr,t}}{6{\cdot}76\,(x^+/N_{Pr,t})^{1/9} + P}\right]^4 + \left[0{\cdot}53835\left(\frac{x^+}{N_{Pr}}\right)^{-1/3}\right]^4\right\}^{1/4} \tag{60}$$

and

$$S_q = \left\{\left[\frac{N_{Pr}/N_{Pr,t}}{6{\cdot}64\,(x^+/N_{Pr,t})^{1/9} + P}\right]^4 + \left[0{\cdot}651\left(\frac{x^+}{N_{Pr}}\right)^{-1/3}\right]^4\right\}^{1/4}. \tag{61}$$

Once again, P is here to be regarded as a function of $(N_{Pr}/N_{Pr,t})$, for example that contained in equation (52).

4. AN OUTLINE OF FURTHER DEVELOPMENTS IN THE THEORY

4.1 *Relevance of the above results*

So far, the line of inquiry into heat transfer through turbulent boundary layers which is represented by references [1], [2], [3] and [4], though more rigorous than that based on the Couette-flow analysis, has not been extended to those problems involving non-uniform fluid properties and mass transfer at finite rates for which Couette-flow analyses have been available for many years. An obstacle to this extension has lain in the apparent prior necessity to possess detailed descriptions of the ϵ_u^+ and ϵ_h^+ functions which are valid in these situations.

This obstacle has been partially removed by the findings described in the earlier parts of the present paper, where it is suggested that a theory adequate for heat-transfer prediction at moderate and large values of x^+ can be built upon two elements: a function $P(N_{Pr}/N_{Pr,t})$, possibly obtained empirically, expressing the "extra resistance" of the region close to the wall; and a solution to equation (1) with ϵ_h^+ assumed equal to $\epsilon_u^+/N_{Pr,t}$ where $N_{Pr,t}$ is a constant, say 0·887. In the latter element, quite rough expressions for ϵ_u^+ may suffice, as witness the success of equations (37) and (39), which are based on power-law expressions for ϵ_u^+, in giving good approximations to the solutions of [3] and [4].

In the following sections, a preliminary discussion will be presented of the way in which the regularities which exist in the exact solutions suggest extensions of the theory to the cases of: rough walls; mass transfer at finite rates; and non-uniform fluid properties. Of course it will be taken for granted that independently derived theories are available for the calculation of the *shear–stress distribution* on the wall under these circumstances; for this forms part of the *data* of the heat-transfer problem, and cannot be deduced from our present theory.

4.2 *Heat transfer across turbulent boundary layers on rough walls*

The solution of the partial differential equation. This element of the theory is relatively easily dealt with: the $S_{T,1}$ and $S_{q,1}$ functions of $(x^+/N_{Pr,t})$ are *precisely the same* as those which are valid for a smooth wall. The reason for the invariance of the solutions is that there is little reason to suppose that roughness affects the ϵ_u^+ function anywhere except at low values of u^+; and we have already seen that precise description of the ϵ_u^+ function is unnecessary close to the wall when N_{Pr} equals $N_{Pr,t}$. Of course, the shear–stress distribution on a rough wall is markedly different from that on a smooth wall, other conditions being equal; however, we have already emphasized that the calculation of this distribution belongs to a different chapter of fluid-dynamic theory which is not to be touched on here.

The P-function. For smooth walls, P has been assumed above to be a function of $N_{Pr}/N_{Pr,t}$ alone. When the wall is rough, however, at least one further parameter may be expected to influence it, namely y_r^+, defined by:

$$y_r^+ \equiv \frac{y_r \sqrt{(\tau_S \rho)}}{\mu} \quad (62)$$

where y_r is the "height of roughness element". The characterization of roughness by such a non-dimensional quantity is of course well known [17]; strictly speaking the geometrical shape and relative spacing of the protruberances on the wall are also important, but some success has been achieved in using y_r^+ alone as the characteristic by defining y_r as a "nominal" rather than "actual" dimension of the roughness.

Two recent publications, by Dipprey and Sabersky [20] and Owen and Thomson [21], fortunately throw much light on the nature of the function $P\,(N_{Pr}/N_{Pr,t}, y_r^+)$. Figure 6 shows values of $[(c_f/2)^{1/2}/N_{St} - (c_f/2)^{-1/2}]$ versus y_r^+ deduced from data presented in these two references. If $N_{Pr,t}$ equals unity, the ordinate has the significance of P; it is seen that P is practically constant for $y_r^+ < 10$, that P rises slowly with y_r^+ when this quantity exceeds about 100, but that P has a minimum value when y_r^+ lies between 10 and 100 for the higher values of Prandtl number.

If the Prandtl number of the turbulent region is not taken as unity, but as a different constant, $N_{Pr,t}$, the ordinate of Fig. 6 has the significance of $N_{Pr,t}\,[P + (N_{Pr,t} - 1)\,(c_f/2)^{-1/2}]$, as is seen by consideration of equation (50), and the subsequent discussion, with t^+ and u^+ evaluated for the main-stream state of the fluid. If $N_{Pr,t}$ is taken as 0·887 and $(c_f/2)$ is estimated to be 0·003, it will be found that the values of the apparent low-y_r^+ asymptotes of Fig. 6 agree fairly well with the P-function of equation (52).

A careful study is needed of all the data available in the literature for heat transfer from rough surfaces before a recommendation can be made for the $P\,(N_{Pr}/N_{Pr,t}, y_r^+)$ function and the associated $N_{Pr,t}$ value (function?) which should be used in design. Nevertheless it is clear that the work of references [20] and [21] forms an already quite advanced starting point, particularly since the authors of both these references have suggested expressions for the asymptotic forms of the P function for high y_r^+, based on the hypothesis of a particular model for the flow induced by the roughness elements. It is to be expected that further research in this field will quickly bear fruit. Of course, it will be necessary to carry out experiments at both higher and lower Prandtl numbers than have been used so far.

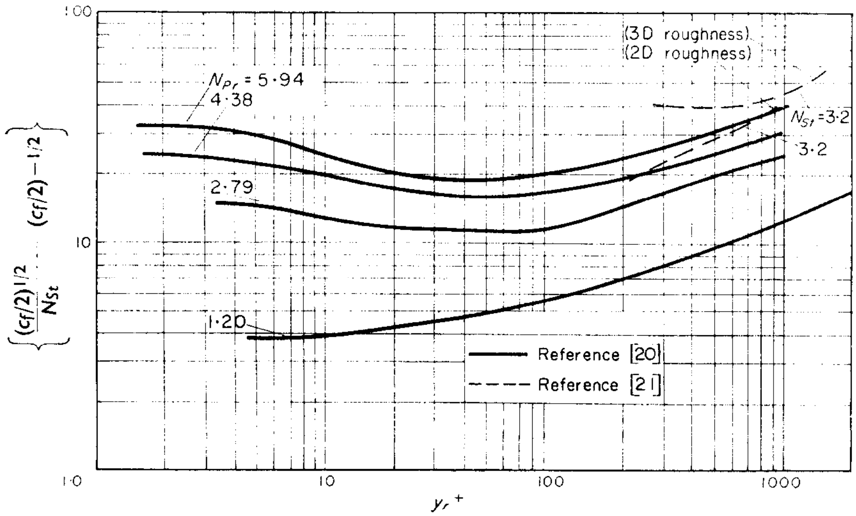

FIG. 6. Experimental data showing the influence of wall roughness on the P-function. If $N_{Pr,t} = 1$, the ordinate is P; otherwise the ordinate is $N_{Pr,t}\,[P + (N_{Pr,t} - 1)\,(c_f/2)^{-1/2}]$.

4.3 *The effect of mass transfer*

The solution of the partial differential equation. When mass transfer occurs at appreciable rates across the wall, as in transpiration cooling, vaporization and condensation, the differential equation (1) requires modification, even for the case of unity $N_{Pr}/N_{Pr,t}$; for variations in shear stress in the immediate vicinity of the wall can no longer be neglected; and a convective component of heat transfer normal to planes of constant u^+ must also be accounted for. It is not proposed to discuss this matter here, except to remark that, at least in the case in which the thermal and velocity boundary layers start from the same line, an adaptation of the method of Spence [15] is capable of yielding a fairly simple solution. The author hopes to demonstrate this in a later publication.

The P-function. We must now expect that P depends on a non-dimensional expression for the mass-transfer rate, as well as on $N_{Pr}/N_{Pr,t}$; such an expression is $\dot{m}'' (\tau_S \rho)^{-1/2}$ where $\dot{m}''$ is the local mass-transfer flux. The influence of this quantity on P can only be established experimentally, but suitable data currently appear to be lacking. This is another field in which a relatively small amount of systematic research would yield rich information.

4.4 *The effect of non-uniform fluid properties*

When the temperature difference across the boundary layer is large, or when the Mach number of flow is high, the values of μ, ρ and N_{Pr} vary significantly, particularly in the laminar sub-layer. It is therefore, to be expected that the thermal resistance of this layer is influenced by at least the first and possibly the second gradients of these properties as functions of distance from the wall. Thus the P function may be expected to vary with the arguments indicated by the following expression, in addition to $N_{Pr}/N_{Pr,t}$, y_r^+ and $\dot{m}'' (\tau_S \rho)^{-1/2}$:

$$P = P\left[\left(\frac{1}{\mu}\frac{\partial u}{\partial u^+}\right)_S, \left(\frac{1}{\rho}\cdot\frac{\partial \rho}{\partial u^+}\right)_S, \left(\frac{1}{\mu}\frac{\partial^2 \mu}{\partial u^{+2}}\right)_S, \left(\frac{1}{\mu}\cdot\frac{\partial^2 \rho}{\partial u^{+2}}\right)_S\right] \quad (63)$$

where subscript S of course denotes values at the wall.

The first two of these are related to the local heat flux by way of the property functions: $\mu(T)$ and $\rho(T)$; they are equal to zero for an adiabatic wall. Possibly the second two are only important in the latter case.

Numerous experimental data are available for heat transfer under the relevant conditions, and it is curious that (so far as the author is aware) no attempt has been made to deduce the above P function from these data. Of course some authors, notably Deissler [12], have implicitly calculated the function by making arbitrary assumptions about the ϵ_u^+ and ϵ_h^+ functions; invariably some sort of agreement between experiment and prediction has been achieved. Since however the comparison has usually been made in terms of Nusselt number at a specified Reynolds number, i.e. at several removes from the initial assumptions, a sound assessment of the validity of the assumptions has been hard to come by. What seems to be required is a comprehensive review of experimental data, plotted for example in the form of P versus $[(1/\mu)(\partial\mu/\partial u^+)]_S$ with other parameters held constant, and with the "theoretical" curves plotted for comparison. Such plots would permit a sounder judgement of whether the "theories" have anything to recommend them. Incidentally, they would also permit purely empirical P functions to be deduced which could thereafter be used for heat-transfer calculations. This too is a task for the future.

Of course, the solutions of the partial differential equation for $N_{Pr}/N_{Pr,t}$ = const. must also be re-examined. However, since the viscous layer is unimportant in these problems, the variation of viscosity will be of no account. Moreover, if Spence's [15] suggestion, viz. that it suffices to replace y by $\int \rho \, dy$, proves to be correct, only minor changes will be needed. The study of the P function is likely to prove more rewarding.

In high-Mach-number problems, it is not sufficient merely to know the Stanton number: the recovery factor, N_{RF} must also be calculated. It is therefore perhaps worth remarking that a reasonable expectation from the above results, which receives partial justification from Spence's paper, is that the recovery factor can be expressed as:

$$\frac{N_{RF}}{c_f/2} = \frac{N_{RF,1}}{c_f/2} + R\,(N_{Pr}/N_{Pr,t}) \tag{64}$$

where $N_{RF,1}$ is the recovery factor which would obtain if N_{Pr} were equal to $N_{Pr,t}$ throughout the boundary layer, and R is a function of $N_{Pr}/N_{Pr,t}$ which accounts for the fact that N_{Pr} is not equal to $N_{Pr,t}$ in the laminar sub-layer. R can easily be calculated for a Prandtl–Taylor sub-layer (i.e. one having a sharp discontinuity between laminar and turbulent regions); since, when compared with N_{RF}, it is multiplied by $c_f/2$, its influence is small. $N_{RF,1}$ can be calculated by Spence's method and for the flat plate is found to be not very different from $N_{Pr,t}$. It is therefore understandable that the recovery factor of a flat plate with a turbulent boundary layer in air is found to be quite close to 0·887 and to be practically independent of Reynolds number.

5. CONCLUSIONS

(a) Exact solutions of the partial-differential equation for heat transfer across a turbulent uniform-property boundary layer have been examined and shown to exhibit regularities, in the region of moderate and large longitudinal distance, which imply that the molecular properties of the fluid exert their influence solely through the agency of a "resistance" to heat transfer at the wall. In this respect the Couette-flow analysis of heat transfer is justified; however, the neglect of the $\partial/\partial x$ terms in the turbulent part of the boundary layer, which is also part of the Couette-flow analysis, is only justified at high Prandtl number, or when the thermal and velocity boundary layers are co-extensive.

(b) Approximate analytical solutions have been provided which give good fits with the exact solutions for moderate and large x^+ [equations (58) and (59)] or for the whole range of x^+ [equations (60) and (61)].

(c) It has been argued that the assumption that the turbulent Prandtl number is unity is no longer tenable, but that the existing exact solutions of the partial differential equation can be generalized so as to hold for any uniform value of $N_{Pr,t}$.

(d) Tentative recommendations have been made for the value of $N_{Pr,t}$ ($= 0{\cdot}887$) and for the form of the P-function [equation (52)].

(e) Methods have been indicated of extending the theory to the cases of a rough wall, mass transfer at an appreciable rate, and non-uniform fluid properties. It is suggested that progress can be made most reliably and swiftly by experimental studies of the influence of various non-dimensional arguments on the P-function; the need for exact knowledge of the variation of ϵ_h with u^+ appears to be correspondingly diminished.

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APPENDIX

Power-law solutions for $N_{Pr} = N_{Pr,t}$

The isothermal wall problem. Equation (54) becomes:

$$\frac{\partial T}{\partial (x^+/N_{Pr})} = \frac{1}{u^+ (1 + \phi)} \frac{\partial^2 T}{\partial u^{+2}}. \quad (A.1)$$

Suppose

$$(1 + \phi) = ab\,(u^+)^{b-1}. \quad (A.2)$$

This corresponds to the velocity profile

$$y^+ = \textstyle\int_0^{u^+} \epsilon_u^+ \, du^+ = a\,(u^+)^b \quad (A.3)$$

i.e. to a power-law velocity profile.

Define:

$$\eta \equiv u^+/(x^+/N_{Pr})^{1/(2+b)} \quad (A.4)$$

and

$$\theta \equiv \frac{T - T_G}{T_S - T_G} \quad (A.5)$$

and suppose:

$$\theta = \theta(\eta). \quad (A.6)$$

The validity of this supposition will be demonstrated by showing that the substitution of (A.4) and (A.5) into the differential equation and boundary conditions eliminates both x^+/N_{Pr} and u^+.

Differentiation of (A.4) yields:

$$\frac{\partial}{\partial x^+} = -\frac{1}{2+b} \frac{\eta}{x^+} \frac{d}{d\eta} \quad (A.7)$$

and

$$\frac{\partial}{\partial u^+} = \frac{1}{(x^+)^{2+b}} \frac{d}{d\eta}. \quad (A.8)$$

So the differential equation becomes:

$$\frac{d^2\theta}{d\eta^2} + \frac{ab}{2+b} \eta^{b+1} \frac{d\theta}{d\eta} = 0 \quad (A.9)$$

which confirms the validity of (A.6) as far as the differential equation is concerned.

The boundary conditions are obviously:

$$\eta = 0 \; : \; \theta = 1 \quad (A.10)$$

$$\eta = \infty : \; \theta = 0. \quad (A.11)$$

The absence of (x^+/N_{Pr}) and u^+ from these conditions completes the justification of (A.6).

A first integration of (A.9) yields:

$$\frac{d\theta}{d\eta} = C \exp\left[\frac{-ab}{(2+b)^2} \eta^{b+2}\right] \quad (A.12)$$

and a second integration yields:

$$\theta = C \int_0^\eta \exp\left[\frac{-ab}{(2+b)^2} \eta^{b+2}\right] d\eta + D \quad (A.13)$$

where C and D are arbitrary constants to be deduced from the boundary conditions. Insertion of these yields:

$$\theta = 1 - \frac{\int_0^\eta \exp\left[\frac{-ab}{(2+b)^2} \eta^{b+2}\right] d\eta}{\int_0^\infty \exp\left[\frac{-ab}{(2+b)^2} \eta^{b+2}\right] d\eta} \quad (A.14)$$

But

$$\int_0^\infty \exp\left[\frac{-ab}{(2+b)^2} \eta^{b+2}\right] d\eta =$$

$$= \left[\frac{(2+b)^2}{ab}\right]^{1/(b+2)} \Gamma\left(\frac{b+3}{b+2}\right). \quad (A.15)$$

Hence we obtain the dimensionless temperature gradient at the wall as:

$$-\left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right)_{\eta=0} = \left[\frac{ab}{(2+b)^2}\right]^{1/(b+2)} \left[\Gamma\left(\frac{b+3}{b+2}\right)\right]^{-1}. \quad \text{(A.16)}$$

Now

$$S_T = \frac{(\partial T/\partial u^+)_S}{T_G - T_S} \quad \text{(A.17)}$$

$$= \left(\frac{x^+}{N_{Pr}}\right)^{-1/(2+b)} \left(\frac{-\mathrm{d}\theta}{\mathrm{d}\eta}\right)_{\eta=0}. \quad \text{(A.18)}$$

Hence

$$\boxed{S_T = \frac{[ab/(2+b)^2]^{1/(b+2)}}{\Gamma\,[(b+3)/(b+2)]} \left(\frac{N_{Pr}}{x^+}\right)^{1/(2+b)}} \quad \text{(A.19)}$$

In the laminar region ϕ equals zero so that $a = b = 1$. Substitution in (A.19) yields:

$$S_T = 0{\cdot}53835\,(x^+/N_{Pr})^{-1/3} \quad \text{(A.20)}$$

This is identical with equation (22) of the main text.

For the turbulent region in which the seventh-power profile is a good approximation, suitable values are: $a = 2{\cdot}412 \times 10^{-7}$, $b = 7$ (this corresponds to the velocity profile $u^+ = 8{\cdot}8(y^+)^{1/7}$). Insertion in (A.19) yields:

$$S_{T,1} = 0{\cdot}1479\,(x^+/N_{Pr,t})^{-1/9} \quad \text{(A.21)}$$

in which the subscript 1 has been added to S_T (as a reminder that $N_{Pr}/N_{Pr,t}$ is taken as unity) and subscript t has been added to N_{Pr} since we are considering the turbulent region. This is the origin of equation (37) of the text.

The problem of the concentrated heat sink (section 2.5 of main text)

Differential equation (A.1) is still valid. Once again (A.2) is substituted in (A.1) and the independent variable η is introduced from (A.4). This time the dependent variable chosen is Ⓗ, defined by:

$$Ⓗ = (T_G - T)\,(x^+/N_{Pr})^{(1+b)/(2+b)} \quad \text{(A.22)}$$

and it is postulated that Ⓗ is a function of η alone.

On transformation to the new variables, (A.1) becomes:

$$\boxed{\frac{\mathrm{d}^2Ⓗ}{\mathrm{d}\eta^2} + \frac{ab}{2+b}\frac{\mathrm{d}}{\mathrm{d}\eta}(Ⓗ\eta^{b+1}) = 0} \quad \text{(A.23)}$$

Boundary conditions are:

$$\eta = 0:\quad \frac{\mathrm{d}Ⓗ}{\mathrm{d}\eta} = 0 \quad \text{(A.24)}$$

$$\eta = \infty:\quad Ⓗ = 0 \quad \text{(A.25)}$$

and in addition:

$$ab\int_0^\infty Ⓗ\eta^{b+1}\,\mathrm{d}\eta = c\mu\,(T_G - T_S)/\dot{q}'. \quad \text{(A.26)}$$

Once again we note that the original variables u^+ and x^+ are absent from the transformed equation and boundary conditions, and conclude that the postulate Ⓗ $=$ Ⓗ(η) is indeed justified.

A first integration of (A.23) yields:

$$\frac{\mathrm{d}Ⓗ}{\mathrm{d}\eta} + \frac{ab}{b+2}Ⓗ\eta^{b+1} = \text{const.} \quad \text{(A.27)}$$

The constant is seen to be zero from boundary condition (A.24).

A second integration yields:

$$Ⓗ = C\exp\left[\frac{-ab}{(b+2)^2}\eta^{b+2}\right] \quad \text{(A.28)}$$

where C is an integration constant to be deduced from the integral condition (A.26). We obtain:

$$\frac{ab\dot{q}'Ⓗ}{c\mu\,(T_G - T_S)} = \frac{\exp\left[\dfrac{-ab}{(b+2)^2}\eta^{b+2}\right]}{\displaystyle\int_0^\infty \eta^b \exp\left[\frac{-ab}{(b+2)^2}\eta^{b+2}\right]\mathrm{d}\eta}. \quad \text{(A.29)}$$

Now

$$\int_0^\infty \eta^b\exp\left[\frac{-ab}{(b+2)^2}\eta^{b+2}\right]\mathrm{d}\eta = $$

$$= \left[\frac{(b+2)^2}{ab}\right]^{(b+1)/(b+2)} \frac{\Gamma\left(\dfrac{2b+3}{b+2}\right)}{(b+1)}. \quad \text{(A.30)}$$

Hence,

$$A_q \equiv \frac{c\mu\,(T_G - T_S)}{\dot{q}'} = \frac{b+1}{(b+2)^2}\left[\frac{(b+2)^2}{ab}\right]^{(b+1)/(b+2)} \left[\Gamma\left(\frac{2b+3}{b+2}\right)\right]^{-1}\left(\frac{N_{Pr,t}}{x^+}\right)^{(b+1)/(b+2)}. \quad (A.31)$$

The problem with uniform $\dot{q}''/(\tau_S)^{1/2}$

S_q may be obtained from:

$$(S_q)^{-1} = \int_0^{(x^+/N_{Pr})} A_q \, d\,(x^+/N_{Pr}). \quad (A.32)$$

which comes from equation (29) of the text.

Hence

$$S_q = \left(\frac{b+2}{b+1}\right)\left[\frac{ab}{(b+2)^2}\right]^{1/(b+2)} \left[\Gamma\left(\frac{2b+3}{b+2}\right)\right]\left(\frac{x^+}{N_{Pr}}\right)^{-1/(b+2)}. \quad (A.33)$$

For the laminar region, $a = b = 1$ as before. Then:

$$S_q = 0{\cdot}651\,(x^+/N_{Pr})^{-1/3} \quad (A.34)$$

which has been presented in the text as equation (24).

For the "seventh-power" velocity profile, $a = 2{\cdot}412 \times 10^{-7}$ and $b = 7$, so that:

$$S_{q,1} = 0{\cdot}1509\,(x^+/N_{Pr,t})^{-1/3} \quad (A.35)$$

wherein the subscript t has been added to N_{Pr} because we are mainly concerned with the turbulent region of the boundary layer, and the subscript 1 has been added to S_q as a reminder that $N_{Pr}/N_{Pr,t}$ is supposed equal to unity.

Résumé—On montre que certaines régularités existent dans les solutions exactes de l'équation aux dérivées partielles du transport de chaleur pour des propriétés uniformes publiée par Gardner et Kestin [3] et Smith et Shah [4]. Ces régularités permettent le développement de formules approchées pour le nombre de Stanton qui sont probablement aussi sûres, comme moyen de prédiction du transport de chaleur, que le sont les solutions basées sur l'intégration numérique.

Les solutions sont généralisées de telle façon qu'elles s'appliquent au cas du "nombre de Prandtl turbulent" constant et différent de 1 et on démontre qu'une valeur au voisinage de 0,887 devrait être utilisée dans un travail futur. On présente une discussion sur la façon selon laquelle la théorie peut-être étendue: aux parois rugueuses; au transport de masse à vitesse finie; et aux propriétés non uniformes du fluide.

Zusammenfassung—Es werden gewisse Regelmässigkeiten gezeigt, die in den exakten Lösungen partieller Differentialgleichungen auftreten für den Wärmeübergang bei einheitlichen Stoffwerten, wie sie von Gardner und Kestin [3] und Smith und Shah [4] veröffentlicht werden. Diese Regelmässigkeiten erlauben die Entwicklung von Näherungsformeln für die Stanton-Zahl, die wahrscheinlich zur Bestimmung des Wärmeübergangs genauso zuverlässig sind, wie die auf numerischer Integration beruhenden Lösungen.

Die Lösungen werden auch auf den Fall ausgedehnt, dass die "turbulente Prandtl-Zahl" eine von Eins verschiedene Konstante ist und es wird empfohlen, in zukünftigen Arbeiten einen Wert nahe 0,887 zu verwenden. In einer Diskussion wird die Erweiterungsmöglichkeit der Theorie auf rauhe Wände, endlichen Stoffaustausch und nicht einheitliche Stoffeigenschaften erörtert.

Аннотация—Показано, что существуют определенные закономерности в точных решениях дифференциального уравнения в частных производных для теплообмена с однородными свойствами, которые были опубликованы Гарднером и Кестиным [3] и Смитом и Шахом [4]. Эти закономерности позволяют вывести приближенные формулы для числа Стантона, которые, повидимому, столь надежны, для расчета теплообмена, как и решения, основанные на численном интегрировании.

Решения обобщены таким образом, что они справедливы для случая, «турбулентное число Прандтля» есть постоянная, отличающаяся от единицы. Доказано, что значение, приблизительно равное 0,887, должно быть использовано в дальнейщей работе. Обсуждаются пути распространения этой теории при шероховатых стенках, массообмене с конечной скоростью и жидкости с неоднородными свойствами.